Complex Variables I
MATH-GA 2450

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Contents

9/12 Lecture

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Schedule: See Syllabus (Brightspace), 10/31 Midterm;
Textbook: Basic Complex Analysis 3rd Edition (1991), Morsden-Hoffman, and Lecture Notes;
Homework: On Brightspace, every Monday, due next Monday 1pm.

0 Motivation

0.1 Holomorphic Functions

Real Analysis: $f : \mathbb{R}^m \to \mathbb{R}^m$ and their differentiability, integration. In this course, we study $f : \mathbb{C} \to \mathbb{C}$ ($\mathbb{R}^2$ + extra structures), with a similar notation of "differentiability" as over $\mathbb{R}^m$, which is called **holomorphic functions**.

- Every holomorphic function is infinitely differentiable ("smooth").
- Every holomorphic function is **analytic** (can be locally developed into a power series, representing this function).

Hence, holomorphic functions is a natural class of functions to study.

0.2 Applications in Real Analysis

- Calculation of certain "complicated integrals":

$$\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} dx = \frac{\pi}{2}, \quad \int_0^{2\pi} \frac{1}{a + \sin(\theta)} d\theta = \frac{2\pi}{\sqrt{a^2 - 1}}, \ a > 0.$$

  $\to$ Residue theorem.

- We will give an elementary proof of the fundamental theorem of algebra: Every non-constant polynomial $p(x)$ with coefficients in $\mathbb{C}$ can be factorized as

$$p(x) = \delta \prod_{j=1}^{m} (x - a_j), \ \delta, a_j \in \mathbb{C}.$$

1 Complex Numbers

We will take from real analysis the existence and uniqueness of a complete, totally ordered field $\mathbb{R}$.

1.1 Definition of Complex Numbers

Historically, $x^2+1=0$ does not have a solution in $\mathbb{R}$. Idea: Assume that some symbol i exists (imaginary unit) s.t. $i^2=-1$ ($i=\sqrt{-1}$) and define complex numbers as formal expressions $x+iy$, $x,y\in\mathbb{R}$.

$$
\begin{aligned}
(x+iy)+(x'+iy') &= x+x'+i(y+y'), \quad \forall x,x',y,y'\in\mathbb{R},\\
(x+iy)(x'+iy') &= xx'-yy'+i(xy'+x'y), \quad \forall x,x',y,y'\in\mathbb{R}.
\end{aligned}
$$

A real x is identified with $x+i\cdot 0$. An expression of the form $0+iy$, $y\in\mathbb{R}$ is called **purely imaginary**. Clearly, $0=0+i\cdot 0$ is the only number to be both real and purely imaginary. If $x+yi\neq 0$,

$$(x+yi)\left(\frac{x}{x^2+y^2}+\frac{y}{x^2+y^2}i\right)=1.$$

Definition 1.1. A **field** F is a set with $+: F\times F\to F$, $\cdot: F\times F\to F$ s.t.

(1) $(F,+)$ is an abelian group, meaning that

  (a) $a+(b+c)=(a+b)+c$,
  (b) $\exists 0_F\in F$: $a+0_F=a$,
  (c) $\forall a\in F$, $\exists(-a)\in F$: $a+(-a)=0_F$,
  (d) $a+b=b+a$;

(2) $(F\backslash\{0_F\},\cdot)$ is an abelian group, meaning that

  (a) $a(bc)=(ab)c$,
  (b) $\exists 1_F\in F\backslash\{0_F\}$: $a\cdot 1_F=a$,
  (c) $\forall a\in F\backslash\{0_F\}$, $\exists(a^{-1})\in F$: $a\cdot(a^{-1})=1_F$,
  (d) $ab=ba$;

(3) $(a+b)c=ac+bc$, $\forall a,b,c\in F$.

Theorem 1.2. We consider $\mathbb{R}^2$ with

$$
\begin{aligned}
(x,y)+(x',y') &= (x+x',y+y'),\\
(x,y)\cdot(x',y') &= (xx'-yy',xy'+yx').
\end{aligned}
$$

Then $\mathbb{C}=\mathbb{R}^2$ is a field with $0_{\mathbb{C}}=(0,0)$, $1_{\mathbb{C}}=(1,0)$, $\mathbb{R}$ is contained in $\mathbb{C}$ as a subfield via the map

$$\phi:\mathbb{R}\to\mathbb{R}^2, \quad x\mapsto(x,0).$$

Every $z=(x,y)$ has a unique representation as $z=x+iy$.

Proof. $\mathbb{R}^2$ is an $\mathbb{R}$-vector space with $0_F=(0,0)$ as 0-vector. Proof for validation of the field will be ignored here. The mapping ϕ fulfills:

$$
\begin{aligned}
&\phi(x+y)=\phi(x)+\phi(y),\\
&\phi(xy)=\phi(x)\phi(y),\\
&\phi(1)=(1,0), \quad \phi(0)=(0,0).
\end{aligned}
$$

Also,

$$(x,y)=(x,0)+(0,y)=(x,0)(1,0)+(0,y)(0,1)=\phi(x)1_{\mathbb{C}}+\phi(y)i,$$

which gives the representation of $z=x+iy$ for $x=(x,y)$. □

Remark 1.3. (1) The inverse of $x=x+yi$ can be intuitively calculated:

$$\frac{1}{z}=\frac{1}{x+yi}=\frac{x-yi}{(x-yi)(x+yi)}=\frac{x-iy}{x^2+y^2}.$$

(2) The field $\mathbb{C}$ cannot be totally ordered. Assume $i>0$, $i\cdot i\cdot i=-i<0$, but $-i>0$ since $(-i)(-i)(-i)=-(-i)<0$.

(3) None of the $\mathbb{R}^n$, $n>3$ can be turned into a field.

1.2 First Properties of Complex Numbers

Definition 1.4. Let $z = x + iy \in \mathbb{C}$. $x = \mathrm{Re}(z) \in \mathbb{R}$ is called **real part of** z, $y = \mathrm{Im}(z) \in \mathbb{R}$ is called **imaginary part of** z. The **modulus** of z is $|z| = \sqrt{x^2 + y^2} \geq 0$. The **complex conjugate** of z is $\overline{z} = x - iy$.

The complex conjugate is taking symmetry about the real axis. Addition follows the parallelogram rule.

Proposition 1.5. Let $z, w \in \mathbb{C}$, then

(1) $\overline{z + w} = \overline{z} + \overline{w}$, $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$,

(2) $|z|^2 = z \cdot \overline{z} = |\overline{z}|^2$,

(3) $z = \overline{z} \iff z \in \mathbb{R}$ ($\mathrm{Im}(z) = 0$),

(4) $\overline{z/w} = \overline{z}/\overline{w}$, $w \neq 0$,

(5) $\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$, $\mathrm{Im}(z) = \frac{1}{2}(z - \overline{z})$,

(6) $\overline{\overline{z}} = z$.

Proof. The proof is trivial by definitions and will be ignored here. □

Proposition 1.6. Let $z, w, z_1, \cdots, z_n, w_1, \cdots, w_n \in \mathbb{C}$, then

(1) $|z| \geq 0$, $|z| = 0 \iff z = 0$,

(2) $-|z| \leq \mathrm{Re}(z) \leq |z|$, $-|z| \leq \mathrm{Im}(z) \leq |z|$,

(3) $|z \cdot w| = |z| \cdot |w|$, $|z/w| = |z|/|w|$, $w \neq 0$,

(4) Triangle inequality: $|z + w| \leq |z| + |w|$, $|z - w| \geq ||z| - |w||$,

(5) Cauchy-Schwarz inequality:

$$\left| \sum_{i=1}^{n} \overline{z_i} w_i \right| \leq \sqrt{\sum_{i=1}^{n} |z_i|^2} \sqrt{\sum_{i=1}^{n} |w_i|^2}.$$

Proof. (1) Trivial.

(2) Trivial.

(3) $|zw|^2 = zw\overline{zw} = z\overline{z}w\overline{w} = |z|^2|w|^2$, and quotient is similar.

(4) $|z + w|^2 = (z + w)(\overline{z + w}) = |z|^2 + |w|^2 + w\overline{z} + z\overline{w} = |z|^2 + |w|^2 + 2\mathrm{Re}(w\overline{z}) \leq |z|^2 + |w|^2 + 2|w\overline{z}| = (|z| + |w|)^2$. Then, $|w| = |z + w - z| \leq |z| + |w - z| \iff |z - w| \geq |w| - |z|$, and changing the order gives $|z - w| \geq |z| - |w|$, so the inverse triangle inequality also holds.

(5) Consider on $\mathbb{C}^n$ the form

$$\langle z, w \rangle = \sum_{i=1}^{n} \overline{z_i} w_i, \quad \langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}.$$

$\langle \cdot, \cdot \rangle$ is positive definite:

$$\langle z, z \rangle = \sum_{i=1}^{n} \overline{z} z \geq 0.$$

$\langle \cdot, \cdot \rangle$ is sesquilinear:

$$\begin{aligned} \langle z + \alpha z', w \rangle &= \langle z, w \rangle + \overline{\alpha} \langle z', w \rangle, \\ \langle z, w + \alpha w' \rangle &= \langle z, w \rangle + \alpha \langle z, w' \rangle, \quad \langle \cdot, \cdot \rangle \text{ is Hermitian.} \end{aligned}$$

Now take $\alpha \in \mathbb{C}$,

$$0 \le \langle w - \alpha z, w - \alpha z \rangle = \langle w, w \rangle - \overline{\alpha} \langle z, w \rangle - \alpha \langle w, z \rangle + |\alpha|^2 \langle z, z \rangle = \langle w, w \rangle + |\alpha|^2 \langle z, z \rangle - 2\mathrm{Re}(\alpha \langle w, z \rangle).$$

Choose

$$\alpha = \frac{\langle w, z \rangle}{\langle z, z \rangle}, \quad z \neq 0,$$

then we have

$$0 \le \langle w, w \rangle + \frac{|\langle w, z \rangle|^2}{\langle z, z \rangle^2} \langle z, z \rangle - 2\mathrm{Re}\left(\frac{|\langle w, z \rangle|^2}{\langle z, z \rangle} \right) = \langle w, w \rangle - \frac{|\langle w, z \rangle|^2}{\langle z, z \rangle}.$$

□

1.3 Polar Coordinates

$\theta = \arg(z)$ is the **argument** of z. For $z = x + yi$, $x = |z|\cos(\theta)$ and $y = |z|\sin(\theta)$. To unambiguously define the argument of $z \neq 0$, we can restrict θ to $(-\pi, \pi]$, called $\mathrm{Arg}(z)$, the **principal value of the argument**. We have the relation

$$\arg(z) = \{\mathrm{Arg}(z) + 2\pi n i,\ n \in \mathbb{Z}\} = \mathrm{Arg}(z) + 2\pi\mathbb{Z}.$$

For $z = r(\cos(\theta) + i\sin(\theta))$, $w = s(\cos(\phi) + i\sin(\phi))$, we have the multiplication as

$$z \cdot w = rs(\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) + i(\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi))) = rs(\cos(\theta + \phi) + i\sin(\theta + \phi)).$$

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(Continued)

To summarize, upon multiplication of two complex numbers, the moduli are multiplied and the arguments are added. The principal value of the argument is shifted modulus 2π back to the interval $(-\pi, \pi]$ if needed.

Proposition 1.7 (De Moivre's formula)**.** If $z \in \mathbb{C}\backslash\{0\}$ with polar coordinate representation

$$z = r(\cos(\theta) + i\sin(\theta)),$$

then

$$z^n = r^n(\cos(n\theta) + \sin(n\theta)).$$

Moreover, the equation $w^n = z$ has exactly n solutions in $\mathbb{C}$ given by

$$w \in \left\{ r^{\frac{1}{n}} \left(\cos\left(\frac{\theta}{n} + \frac{2\pi h}{n} \right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi h}{n} \right) \right);\ h \in \{0, \cdots, n-1\} \right\}.$$

Proof. For the first part, just iterate n times to get the result. For the second part, see notes. Check that

$$\left(r^{\frac{1}{n}} \left(\cos\left(\frac{\theta}{n} + \frac{2\pi h}{n} \right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi h}{n} \right) \right) \right)^n = r(\cos(\theta + 2\pi h) + i\sin(\theta + 2\pi h)) = z.$$

□

Remark 1.8. (1) Looking at de Moivre's formula for say $n = 3$,

$$\cos(3\theta) + i\sin(3\theta) = (\cos(\theta) + i\sin(\theta))^3 = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta) + (3\cos^2(\theta)\sin(\theta) - \sin^3(\theta))i.$$

Hence, we can get some nice trigonometric identities by comparing the real and imaginary parts respectively on the two sides of the equation.

(2) Avoid notations such as $\sqrt{-1}$ or $\sqrt[3]{1+i}$, ...

(3) Complex conjugation in polar coordinates: if $z = r(\cos(\theta) + i\sin(\theta))$, then

$$z = r(\cos(\theta) - i\sin(\theta)) = r(\cos(-\theta) + i\sin(-\theta)).$$

Hence, $\mathrm{Arg}(\overline{z}) = -\mathrm{Arg}(z)$, $z \in \mathbb{C}\backslash\mathbb{R}$.

1.4 Elementary Complex Functions

Polynomials and rational functions.

$$P : \mathbb{C} \to \mathbb{C} \qquad z \mapsto a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \text{ where } a_0, \cdots, a_n \in \mathbb{C},\ a_n \neq 0.$$

An expression of this form is a polynomial of degree $n \in \mathbb{N}_0$. We will see: polynomials of degree $n \geq 1$ have (counting multiplicities) exactly n zeros.

Proposition 1.9. If z is a zero of a polynomial P with $a_0, \cdots, a_n \in \mathbb{R}$, then $\overline{z}$ is also a zero.

Proof.

$$a_n z^n + \cdots + a_0 = P(z) = 0 \implies \overline{a_n z^n + \cdots + a_0} = \overline{0} = 0.$$

Hence, $P(\overline{z}) = \overline{a_n}\,\overline{z}^n + \cdots + \overline{a_0} = 0$. □

We will also encounter rational functions:

$$f : \mathbb{C} \setminus \{z_1, \cdots, z_n\} \to \mathbb{C} \qquad z \mapsto \frac{P(z)}{Q(z)},$$

where P, Q are polynomials and $z_1, \cdots, z_n$ are the zeros of Q.

Exponential and trigonometric functions. Recall:

$$\exp : \mathbb{R} \to \mathbb{R} \qquad x \mapsto \sum_{h=0}^{\infty} \frac{x^h}{h!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Heuristic: Formally insert iy, $y \in \mathbb{R}$ into exp, so

$$\begin{aligned}
\exp(iy) &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \frac{(iy)^6}{6!} + \cdots \\
&= 1 + iy - \frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} + \frac{iy^5}{5!} - \frac{y^6}{6!} - \cdots \\
&= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right) = \cos(y) + i\sin(y).
\end{aligned}$$

This is called the Euler formula.

Definition 1.10. We define the **complex exponential function**

$$\exp : \mathbb{C} \to \mathbb{C} \qquad x + yi \mapsto \exp(x)(\cos(y) + i\sin(y)).$$

Proposition 1.11. (1) $\forall z, w \in \mathbb{C}$: $\exp(z+w) = \exp(z) \cdot \exp(w)$,

(2) $\exp(\mathbb{C}) = \{\exp(z);\ z \in \mathbb{C}\} = \mathbb{C} \setminus \{0\}$,

(3) $\exp(yi) = \cos(y) + i\sin(y)$, $|\exp(x+yi)| = \exp(x)$, $x, y \in \mathbb{R}$,

(4) $\exp(2\pi hi + z) = \exp(z)$, $z \in \mathbb{C}$, $h \in \mathbb{Z}$, and $\exp(z) = 1$ if and only if $z \in 2\pi i\mathbb{Z}$,

(5) $\overline{\exp(x+yi)} = \exp(x - yi)$, $x, y \in \mathbb{R}$, $\overline{\exp(z)} = \exp(\overline{z})$, $z \in \mathbb{C}$.

Proof. (1) Trivial by some simple trigonometric identities.

(2) Take $w \neq 0$, try to find $z \in \mathbb{C}$ such that $\exp(z) = w$. Note:

$$w = |w| \cdot \frac{w}{|w|}, \text{ where } |w| \in \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\} = (\cos, \sin)(\mathbb{R}).$$

Hence, for $w \neq 0$ we can set $x = \log|w|$, and find $y \in \mathbb{R}$ such that

$$\cos(y) + i\sin(y) = \frac{w}{|w|}.$$

(3) Trivial.

(4) Trivial by (1).

(5) Trivial by some conjugate identities. □

Particular values to keep in mind:

$$\exp\left(\frac{\pi}{2}i\right) = i, \qquad \exp(\pi i) = -1,$$
$$\exp\left(\frac{3\pi}{2}i\right) = -i, \qquad \exp(2\pi i) = 1.$$

Definition 1.12. Define the **complex trigonometric functions**:

$$\sin : \mathbb{C} \to \mathbb{C} \qquad \sin(z) = \frac{1}{2i}(\exp(iz) - \exp(-iz)),$$
$$\cos : \mathbb{C} \to \mathbb{C} \qquad \cos(z) = \frac{1}{2}(\exp(iz) + \exp(-iz)).$$

This definition is motivated by: $y \in \mathbb{R}$, then $\exp(iy) + \exp(-iy) = 2\cos(y)$, etc.

Proposition 1.13. (1) $\sin^2(z) + \cos^2(z) = 1$, $\forall z \in \mathbb{C}$,

(2) Euler formula: $\exp(iz) = \cos(z) + i\sin(z)$, $\forall z \in \mathbb{C}$,

(3) $\sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$, $\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$.

Proof. Trivial by definitions and simple computations. □

Logarithms.

Definition 1.14. Let $w \in \mathbb{C}\setminus\{0\}$, a solution to $\exp(z) = w$ is called a **logarithm** of w, written as $z = \log(w)$.

Proposition 1.15. Every $w \in \mathbb{C}\setminus\{0\}$ has countably many logarithms given by:

$$\log(w) = \log(|w|) + i\arg(w) = \log(|w|) + \{i(\operatorname{Arg}(w) + 2\pi n);\ n \in \mathbb{Z}\}.$$

Proof. Let z be in the set above, then

$$\exp(z) = \exp(\log(|w|))\exp(i\cdot\operatorname{Arg}(w) + 2\pi in) = |w|\cdot\frac{w}{|w|} = w.$$

Now, let us assume $z = x + iy$ is a logarithm of w:

$$\exp(z) = \exp(x)(\cos(y) + i\sin(y)) = |w|(\cos(y) + i\sin(y)),\ \theta = \operatorname{Arg}(w).$$

This implies that $x = \log(|w|)$, $\theta - y \in 2\pi\mathbb{Z}$. □

Definition 1.16. For $w \in \mathbb{C}\setminus\{0\}$, we set

$$\operatorname{Log}(w) = \log(|w|) + i\cdot\operatorname{Arg}(|w|),$$

with $\operatorname{Arg}(w) \in (-\pi, \pi]$, the **principal value of the logarithm of** w.

The function $\operatorname{Log} : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ that attains values in $\mathbb{R}\times(-\pi,\pi] \subseteq \mathbb{C}$ is then called the **principle branch of the logarithm**. At this point, one should recall that the choice $\operatorname{Arg} \in (-\pi,\pi]$ was somewhat arbitrary, and one might have translated this interval to $\operatorname{Arg} \in (y_0, y_0 + 2\pi]$ for any $y_0 \in \mathbb{R}$. Naturally, there exists a proper function $\operatorname{Log} : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ whose range is

$$A_{y_0} = \{x + iy;\ y \in (y_0, y_0 + 2\pi]\} = \mathbb{R}\times(y_0, y_0 + 2\pi],$$

which is the **branch of the logarithm lying in** A_{y_0}.

Proposition 1.17. The restriction $\exp|_A$ of the exponential function to $A = A_{-\pi} = \{z = x + yi;\ -\pi < y \leq \pi\} \subseteq \mathbb{C}$ gives a bijection from A to $\mathbb{C}\setminus\{0\}$, with inverse given by $w \mapsto \operatorname{Log}(w)$.

Proof. For the injectivity of the restricted exponential, let $\exp(z) = \exp(w)$ for $z, w \in A$, then we have $\exp(z-w) = 1$. Hence, $z - w \in 2\pi i\mathbb{Z}$. Moreover, since $|\mathrm{Im}(z) - \mathrm{Im}(w)| < 2\pi$, we must have $z = w$, so injectivity is done. The surjectivity is also clear since $(\cos, \sin)((-\pi, \pi]) = (\cos, \sin)(\mathbb{R}) = \mathbb{S}^1$ and by some proposition above. Finally, the function Log maps $\mathbb{C}\setminus\{0\}$ to A, and is the inverse of $\exp|_A$ by definition. $\square$

Remark 1.18. (1) We stress again that due to periodicity, one cannot hope for an inverse of $\exp : \mathbb{C} \to \mathbb{C}$ (in the same way as, for instance, $\sin : \mathbb{R} \to \mathbb{R}$ has no inverse). So care is required even for "natural" looking expressions. In general, $\mathrm{Log}(\exp(z)) \neq z$ (unless $\mathrm{Im}(z) \in (-\pi, \pi]$), for instance

$$\mathrm{Log}(\exp(2 + 3\pi i)) = \mathrm{Log}(-\exp(2)) = 2 + \pi i.$$

(2) In the same way, the rules for logarithms must be adapted to reflect this issue: For instance, it is true that for $z, w \in \mathbb{C}\setminus\{0\}$,

$$\log(zw) = \log(z) + \log(w),$$

which is comprehended as an equality between sets. However one only has

$$\mathrm{Log}(zw) - \mathrm{Log}(z) - \mathrm{Log}(w) \in 2\pi i\mathbb{Z}.$$

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2 Continuity

2.1 Complex Sequences and Series

Definition 2.1. A sequence $(z_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is **bounded** if

$$\exists C > 0 : \quad |z_n| \leq C, \ \forall n \geq 1.$$

It is called **convergent** with **limit** $z \in \mathbb{C}$ if

$$\forall \epsilon > 0, !!!$$

Proposition 2.2. (1) The limit of a sequence (if exists) is unique.

(2) Convergent sequences are bounded.

(3) If $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} w_n = w$, then

$$\lim_{n\to\infty}(z_n + w_n) = z + w,$$
$$\lim_{n\to\infty}(z_n \cdot w + n) = z \cdot w.$$

(4) If $\lim_{n\to\infty} z_n \neq 0$, then

$$\lim_{n\to\infty} \frac{1}{z_n} = \frac{1}{z}.$$

(5) If $\lim_{n\to\infty} z_n = z$, then

$$\lim_{n\to\infty} \mathrm{Re}(z_n) = \mathrm{Re}(z); \qquad \lim_{n\to\infty} |z_n| = |z|;$$
$$\lim_{n\to\infty} \mathrm{Im}(z_n) = \mathrm{Im}(z); \qquad \lim_{n\to\infty} \overline{z_n} = \overline{z}.$$

(6) If $\lim_{n\to\infty} \mathrm{Re}(z_n) = \mathrm{Re}(z)$ and $\lim_{n\to\infty} \mathrm{Im}(z_n) = \mathrm{Im}(z)$, then

$$\lim_{n\to\infty} z_n = z.$$

Proof. (1) Assume z and z' are both limits of z_n, then given $\epsilon > 0$, $\exists N, N \in \mathbb{N}$, such that

$$|z_n - z| < \epsilon,\ \forall n \geq Nl \quad |z_n - z'| < \epsilon,\ \forall n \geq N'.$$

Taking $n \geq \max\{N, N'\}$, we have that

$$|z - z'| \leq |z_n - z| + |z_n - z'| < 2\epsilon.$$

By arbitrariness of ϵ we conclude that the limit is unique.

(2) Fix $\epsilon = 1$, then $\forall n > N(\epsilon)$, $|z - z_n| < 1$. We can write

$$|z_n| = |z_n + z - z| \leq |z_n - z| + |z| < |z| + 1,\ \forall n > N(\epsilon).$$

Hence, we can conclude that

$$|z_n| \leq \max\left\{\max_{1\leq i\leq N(\epsilon)-1} |z_i|, |z| + 1\right\}.$$

(3) For $\epsilon > 0$, $\exists N(\epsilon), M(\epsilon) \in \mathbb{N}$, such that

$$|z_n - z| \leq \frac{\epsilon}{2},\ \forall n \geq N(\epsilon); \quad |w_n - w| \leq \frac{\epsilon}{2},\ \forall n \geq M(\epsilon).$$

Then,

$$|z_n + w_n - (z + w)| \leq |z_n - z| + |w_n - w| \leq \epsilon,\ \forall n > \max\{M(\epsilon), N(\epsilon)\}.$$

Also, $\forall n \in \mathbb{N}$, we have

$$|z_n w_n - zw| \leq |z_n w_n - z_n w| + |z_n w - zw| \leq C|w_n - w| + |w||z_n - z|,$$

where C is the upper bound of $|z_n|$ since it is convergent and by (2). Now $\exists N'(\epsilon), M'(\epsilon) \in \mathbb{N}$, such that

$$|z_n - z| \leq \frac{\epsilon}{2(1 + |w|)},\ \forall n \geq N'(\epsilon); \quad |w_n - w| \leq \frac{\epsilon}{2(1 + C)},\ \forall n \geq M'(\epsilon).$$

Then,

$$|z_n w_n - zw| < \frac{\epsilon C}{2(1 + C)} + \frac{\epsilon|w|}{2(1 + |w|)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\ \forall n \geq \max\{M'(\epsilon), N'(\epsilon)\}.$$

(4) We assume (5), then $|z_n| \to |z|$ and $\overline{z_n} \to \overline{z}$ as $n \to \infty$. Then

$$\frac{1}{z_n} = \frac{\overline{z_n}}{z_n \cdot \overline{z_n}} = \frac{\overline{z_n}}{|z_n|^2} \to \frac{\overline{z}}{|z|^2} = \frac{1}{z} \quad \text{as } n \to \infty.$$

(5) We have

$$\begin{aligned}
&|\mathrm{Re}(z_n) - \mathrm{Re}(z)| = |\mathrm{Re}(z_n - z)| \leq |z_n - z|,\\
&|\mathrm{Im}(z_n) - \mathrm{Im}(z)| = |\mathrm{Im}(z_n - z)| \leq |z_n - z|,\\
&||z_n| - |z|| \leq |z_n - z| \text{ (reverse triangle inequality)},\\
&|\overline{z_n} - \overline{z}| = |\overline{z_n - z}| = |z_n - z|,
\end{aligned}$$

and the rest of the proof is trivial.

(6) We have

$$|z_n - z| \leq |\mathrm{Re}(z_n) - \mathrm{Re}(z)| + |\mathrm{Im}(z_n) - \mathrm{Im}(z)|,$$

and the rest of the proof is trivial. ∎

Definition 2.3. A sequence of complex numbers $(z_n)_{n\in\mathbb{N}}$ is called a **Cauchy sequence** if

$$\forall \epsilon > 0: \quad \exists N(\epsilon) \in \mathbb{N},\ |z_n - z_m| < \epsilon,\ \forall m, n \geq N(\epsilon).$$

Proposition 2.4. A sequence of complex numbers $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if and only if it converges.

Proof. ($\Longleftarrow$) Assume $(z_n)_{n\in\mathbb{N}}$ converges to $z \in \mathbb{C}$. Given $\epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$, such that $|z_n - z| < \frac{\epsilon}{2}$, $\forall n \in \mathbb{N}$. Then

$$|z_n - z_m| \leq |z_n - z| + |z_m - z| \leq \epsilon,\ \forall n, m > N(\epsilon).$$

($\Longrightarrow$) Recall: We assume that $\mathbb{R}$ is complete, meaning that every Cauchy sequence in $\mathbb{R}$ converges. Now, since (z_n) is a Cauchy sequence,

$$|\mathrm{Re}(z_n) - \mathrm{Re}(z_m)| < |z_n - z_m| \leq \epsilon, \quad |\mathrm{Im}(z_n) - \mathrm{Im}(z_m)| \leq |z_n - z_m| < \epsilon.$$

Hence, $(\mathrm{Re}(z_n))_{n\in\mathbb{N}}$ and $(\mathrm{Im}(z_n))_{n\in\mathbb{N}}$ are both Cauchy sequences in $\mathbb{R}$. By completeness of $\mathbb{R}$, they converge to some x and y respectively, so (z_n) will converge to $x + iy$. □

Definition 2.5. An **infinite series** $\sum_{v=1}^{\infty} z_v$ with $z_v \in \mathbb{C}$ is understood as the sequence $(S_n)_{n\in\mathbb{N}}$ of partial sums, defined as

$$S_n = \sum_{v=1}^{\infty} z_v.$$

If $(S_n)_{n\in\mathbb{N}}$ converges and has limit $S \in \mathbb{C}$, we say the series is **convergent** and write $S = \sum_{v=1}^{\infty} z_v$.

Remark 2.6. (1) An example for a convergent series is the **geometric series**

$$\sum_{v=0}^{\infty} z^v, \quad |z| < 1.$$

We have

$$(1-z)\sum_{v=0}^{n} z^v = (1 + z + \cdots + z^n) - (z + \cdots + z^n + z^{n+1}) = 1 - z^{n+1},$$

and since $|z|^n \to 0$ as $n \to \infty$, we have that

$$\sum_{v=0}^{\infty} z^v = \frac{1}{1-z}, \quad |z| < 1.$$

(2) Complex power series are series of the form $\sum_{v=1}^{\infty} a_v z^v$ with $(a_v)_{v\in\mathbb{N}} \subseteq \mathbb{C}$. These will play a crucial role in later chapters.

(3) If $\sum_{v=1}^{\infty} z_v$ converges, then $(z_n)_{n\in\mathbb{N}}$ converges to 0. Indeed, the sequence of partial sums $(s_n)_{n\in\mathbb{N}}$ is then a Cauchy sequence and

$$|z_n| = |s_{n+1} - s_n| \leq \epsilon,\ \forall n > N(\epsilon).$$

If $\sum_{v=1}^{\infty} |z_v|$ converges (in $\mathbb{R}$), then $(z_n)_{n\in\mathbb{N}}$ also converges, and we say that $\sum_{v=1}^{\infty} z^v$ is **absolutely convergent**. Indeed, use the fact that

$$|s_{n+k} - s_n| = \left|\sum_{v=n+1}^{n+k} z_v\right| \leq \sum_{v=n+1}^{n+k} |z_v|.$$

2.2 Topology of $\mathbb{C}$, Continuous Functions

Definition 2.7. (1) Let $r > 0$, $z_0 \in \mathbb{C}$. We call

$$D(z_0, r) = \{z \in \mathbb{C} : \ |z - z_0| < r\}$$

the **(open) disk around** z_0 **with radius** r. Also,

$$\dot{D}(z_0, r) = D(z_0, r) \setminus \{z_0\}$$

is called the **deleted (open) disk around** z_0 **with radius** r.

(2) Let $A \subseteq \mathbb{C}$, a point $z_0 \in A$ is called an **interior point of** A if

$$\exists r > 0 : \quad D(z_0, r) \subseteq A.$$

(3) A set A is called **open** if every point $z_0 \in A$ is an interior point.

(4) A set A such that $A^C := \mathbb{C} \setminus A$ is open is called **closed**.

Proposition 2.8. (1) $\varnothing$, $\mathbb{C}$ are both closed and open.

(2) The intersection of arbitrarily many closed sets is closed, and the union of arbitrarily many open sets is open.

(3) The union of finitely many closed sets is closed, and the intersection of finitely many open sets is open.

Definition 2.9. Let $A \subseteq \mathbb{C}$. We define the **interior** of A by

$$\mathring{A} = \bigcup_{\mathscr{U} \subseteq A, \mathscr{U} \text{ open}} \mathscr{U}$$

and the **closure** of A by

$$\overline{A} = \bigcap_{\mathscr{C} \supseteq A, \mathscr{C} \text{ closed}} \mathscr{C}.$$

Finally, the **boundary** of A is defined as

$$\partial A = \overline{A} \setminus \mathring{A}.$$

Definition 2.10. (1) A subset $A \subseteq \mathbb{C}$ is called **bounded**, if there is $C \leq 0$ such that for every $z \in A$, one has $|z| \leq C$.

(2) We say that $A \in \mathbb{C}$ is **compact**, if every open cover of A admits a finite subcover, that is if for any collection $\{U_\alpha\}_{\alpha \in \mathscr{A}}$ with $A \subseteq \bigcup_{\alpha \in \mathscr{A}} U_\alpha$ and U_α open, there are finitely many $\alpha_1, \cdot, \alpha_n \in \mathscr{A}$ such that $A \subseteq \bigcup_{j=1}^{n} U_{\alpha_j}$.

Proposition 2.11. The following conditions for a subset $A \subseteq \mathbb{C}$ are equivalent:

(1) A is compact.

(2) A is closed and bounded. (Recall Heine-Borel Theorem in $\mathbb{R}$.)

(3) Every sequence $(z_n)_{n \in \mathbb{N}}$ has a convergent subsequence with limit in A.

Definition 2.12. Let $A \subseteq \mathbb{C}$ and let $f : A \to \mathbb{C}$. Let $z_0 \in \mathbb{C}$ be an **accumulation point** of A (*i.e.*, a point such that $\forall r > 0$, there are finitely many points in $D(z_0, r) \cap A$). We say that f has a **limit** w at z_0, denoted

$$\lim_{z \to z_0} f(z) = w,$$

if $\forall \epsilon > 0$, $\exists \delta > 0$, such that

$$|f(z) - w| < \epsilon, \ \forall z \in \dot{D}(z_0, \delta) \cap A.$$

Remark 2.13. Limits are (if they exists) unique.

Definition 2.14. Let $A \subseteq \mathbb{C}$ and let $f : A \to \mathbb{C}$. We say that f is **continuous** at $z_0 \in A$ if

$$\forall \epsilon > 0 : \quad \exists \delta > 0,\ |f(z) - f(z_0)| < \epsilon,\ \forall z \in D(z_0, \delta) \cap A.$$

Note that if $z_0 \in A$ is an **isolated point** (*i.e.*, not an accumulation point), then every f is continuous at z_0.

Lemma 2.15. Let $A \subseteq \mathbb{C}$, $f : A \to \mathbb{C}$ a function, and $z_0 \in A$ and accumulation point. Then the following are equivalent:

(1) f is continuous at z_0.

(2) $\lim_{z\to z_0} f(z) = f(z_0)$.

(3) For any sequence $(z_n)_{n\in\mathbb{N}} \subseteq A$ with $\lim_{n\to\infty} z_n = z_0$, we have

$$\lim_{n\to\infty} f(z_n) = f(z_0)$$

Proof. ($1 \implies 2$) Trivial.
($2 \implies 3$) Let $\epsilon > 0$, and suppose $(z_n)_{n\in\mathbb{N}} \subseteq A$ converges to z_0. There exists $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$, provided $|z - z_0| < \delta$. Then choose $N \in \mathbb{N}$ (depending on this δ) such that $|z_n - z_0| < \delta$ for every $n \geq N$.
($3 \implies 1$) By contradiction, f not continuous at z_0 means that

$$\exists \epsilon > 0 : \quad \forall n \geq 1,\ \exists z_n \in A \cap D\left(z_0, \frac{1}{n}\right),\ |f(z_n) - f(z_0)| \geq \epsilon.$$

The sequence $(z_n)_{n\in\mathbb{N}} \subseteq A$ clearly converges to z_0, but $(f(z_n))_{n\in\mathbb{N}}$ does not converge to $f(z)$, giving the desired contradiction. □

Proposition 2.16. Let $A \in \mathbb{C}$ and $f, g : A \to \mathbb{C}$ be two functions that are continuous at $z_0 \in A$.

(1) The functions $f + g$ and $f \cdot g$, defined as

$$f + g : \begin{cases} A \to \mathbb{C} \\ z \mapsto f(z) + g(z) \end{cases} \qquad \text{and} \quad f \cdot g : \begin{cases} A \to \mathbb{C} \\ z \mapsto f(z)g(z) \end{cases},$$

are continuous at z_0.

(2) The functions $\mathrm{Re}f : z \mapsto \mathrm{Re}(f(z))$, $\mathrm{Im}f : z \mapsto \mathrm{Im}(f(z))$, $\overline{f} : z \mapsto \overline{f(z)}$ and $|f| : z \mapsto |f(z)|$ (all defined on A) are continuous at z_0.

(3) If $g(z_0) \neq 0$, there is $\delta > 0$ such that $g(z) \neq 0$ for $z \in A \cap D(z_0, \delta)$ and the function f/g, defined as

$$\frac{f}{g} : \begin{cases} A \cap D(z_0, \delta) \to \mathbb{C} \\ z \mapsto \frac{f(z)}{g(z)} \end{cases},$$

is continuous at z_0.

(4) If $h : B \to \mathbb{C}$ is a function defined on $B \supseteq f(A)$ and is continuous at $f(z_0)$, then the composition $h \circ f : A \to \mathbb{C}$ is continuous at z_0 as well.

Proof. The proofs are trivial. The only thing to notice is that in (3), we may want to find a lower bound for $|g(z)|$, which can be $\frac{|g(z_0)|}{2}$ by taking $\epsilon = \frac{|g(z_0)|}{2} > 0$ in the continuity condition. □

Remark 2.17. (1) Polynomials are continuous on $\mathbb{C}$ and rational functions are continuous away from their poles. The functions exp, sin, and cos are also continuous on $\mathbb{C}$.

(2) The functions defined on $\mathbb{C}\setminus\{0\}$, $z \mapsto \mathrm{Arg}(z)$ and $z \mapsto \mathrm{Log}(z)$ are not continuous. Consider for instance the sequence $z_n = \exp(\theta_n i)$, where $\theta_n = \pi - \frac{1}{n}$ when n is odd, and $\theta_n = -\pi + \frac{1}{n}$ when n is even. Clearly, $\lim_{n\to\infty} z_n = -1$, but $\mathrm{Arg}(z_n) = \theta_n$ does not converge. The same happens for Log. Nevertheless, both functions are continuous when restricted to $\mathbb{C}_- = \mathbb{C}\setminus\{(x, 0);\ x \leq 0\}$.

(3) Let $A \subseteq \mathbb{C}$ and $f : A \to \mathbb{C}$ be continuous. If $K \subseteq A$ is compact, then $f(K)$ is compact as well. Moreover, f attains its maximum and minimum on K, meaning that there is a points $\hat{z} \in K$ with $f(\hat{z}) = \sup_{z \in K} |f(z)|$ (and similar for the minimum).

Definition 2.18. Let $A \subseteq \mathbb{C}$. Then,

(1) A is called **connected** if it cannot be written as a disjoint union of two non-empty, relatively open subsets A_1 and A_2. [1]

(2) A is called **path-connected** if for every two points $z, w \in A$, one can find a continuous map $\gamma : [0, 1] \to A$ with $\gamma(0) = z$ and $\gamma(1) = w$.

(3) A is called a **domain** if it is both open and connected.

Proposition 2.19. (1) Every path-connected set $A \in \mathbb{C}$ is connected.

(2) If A is a domain, then it is also path-connected. In fact, the path $\gamma : [0, 1] \to A$ can be taken to be continuously differentiable (as a curve in $\mathbb{R}^2$, that is, componentwise continuously differentiable).

(3) The image of a connected set A under a continuous map is again connected.

Proof. (1) Suppose C is a path-connected set and $\varnothing \neq D \subseteq C$ is both open and closed relative to C. If $C \neq D$, then there exist $z_1 \in D$ and $z_2 \in C \backslash D$. Let $\gamma : [a, b] \to C$ be a continuous path joining z_1 to z_2. Let $B = \gamma^{-1}(D)$, then $B \subseteq [a, b]$ by continuity. Since $z_1 = \gamma(a) \in D$, $a \in B$, so $B \neq \varnothing$. Similarly, $[a, b] \backslash B \neq \varnothing$ since it contains b. This argument shows that it is sufficient to prove the theorem for the case of an interval $[a, b]$. We thus need to establish that intervals on a real line are connected. A proof uses the least upper bound property. Let $x = \sup B$, then $x \in B$ since B is closed. Meanwhile, since B is open, there exists a small disk $D(x, \delta) \subseteq B$. Thus, for some $\epsilon > 0$, $x + \epsilon \in B$. Thus x cannot be the least upper bound. The contradiction shows that such a set B cannot exist.

(2) Let $a \in A$. If $z_0 \in A$, then since A is open, there exists a small disk $D(z_0, \delta) \subseteq A$. By combining a path from a to z_0 with one from z_0 to z that stays in this disk, we see that z_0 can be connected to a by a differentiable path if and only if the same is true for every point z in $D(z_0, \delta)$. This shows that both the sets

$$A_1 = \{z \in \mathbb{C};\ z \text{ can be connected to } a \text{ by a differentiable path}\}$$

and

$$A_2 = \{z \in \mathbb{C};\ z \text{ cannot be connected to } a \text{ by a differentiable path}\}$$

are open. Since A is connected, either A_1 or A_2 must be empty. Obviously, it must be $A_2 = \varnothing$.

(3) Let f be a continuous map defined on A. If U and V are open sets that disconnect $f(A)$, then $f^{-1}(U)$ and $f^{-1}(V)$ are open sets that disconnect A, which leads to a contradiction.

□

2.3 Extended Complex Plane $\overline{\mathbb{C}}$, Riemann Sphere

We introduce the concept of the **point at infinity**, denoted by ∞ (which is not in $\mathbb{C}$) as an additional point and join it to $\mathbb{C}$. This is called the **extended complex plane** $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. On $\overline{\mathbb{C}}$, we introduce the following operations:

$$\begin{aligned} z + \infty &= \infty, & z \cdot \infty &= \infty,\ z \neq 0, \\ \infty + \infty &= \infty, & \infty \cdot \infty &= \infty, \\ \frac{z}{\infty} &= 0, & \frac{z}{0} &= \infty,\ z \neq 0. \end{aligned}$$

Definition 2.20. A subset $A \subseteq \overline{\mathbb{C}}$ is called **open**, if the following holds:

(1) $A \cap \mathbb{C}$ is open;

(2) If $\infty \in A$, there exists $K > 0$, such that $D(\infty, K^{-1}) = \{z \in \mathbb{C};\ |z| > K\} \cup \{\infty\} \subseteq A$.

[1] $B \subseteq A$ is **relatively open** if there exists an open set $U \in \mathbb{C}$ with $B = U \cap A$.

Limits involving ∞ are then defined accordingly:

- For a sequence $(z_n)_{n\in\mathbb{N}} \subseteq \overline{\mathbb{C}}$, $\lim_{n\to\infty} z_n = \infty$ means that for every $K > 0$, there exists $N \in \mathbb{N}$, such that $|z_n| > K$ for every $n \geq N$.
- For a function f defined on a set containing some $\dot{D}(\infty, K^{-1}) = \{z \in \mathbb{C};\ |z| > K\}$, we write $\lim_{z\to\infty} f(z) = w$ if for any $\epsilon > 0$, there exists $K > 0$, such that $|f(z) - w| < \epsilon$ whenever $|z| > K$.
- Finally, for a function f defined on $A \subseteq \overline{\mathbb{C}}$ with accumulation point $z_0 \in \mathbb{C}$, we write $\lim_{z\to z_0} f(z) = \infty$ if for every $K > 0$, there exists $\delta > 0$, such that $|f(z)| > K$ for all $z \in \dot{D}(z_0, \delta) \cap A$.

A geometric model to visualize $\overline{\mathbb{C}}$ is given by the **Riemann sphere**

$$S^2 = \left\{(x, y, s) \in \mathbb{R}^3;\ x^2 + y^2 + s^2 = 1\right\} = \left\{(z, s) \in \mathbb{C} \times \mathbb{R};\ |z|^2 + s^2 = 1\right\},$$

in which the xy-plane is supposed to represent the complex plane, as is shown in Figure 1. If we connect the north pole $N = (0, 1)$ with a point $(w, 0) \in \mathbb{C} \times \{0\}$ by a straight line, it intersects S^2 in exactly two points, N and $\tau(w)$. We call the inverse function $\sigma : S^2 \setminus \{N\} \to \mathbb{C}$ the **stereographic projection**. If move the second intersection point (z, s) of the line with the sphere closer to N, we see that the corresponding point $(\sigma(z, s), 0)$ moves further away from the origin, *i.e.*, $|\sigma(z, s)|$ becomes larger and larger. We thus naturally define N (or rather its image) with the point at infinity.

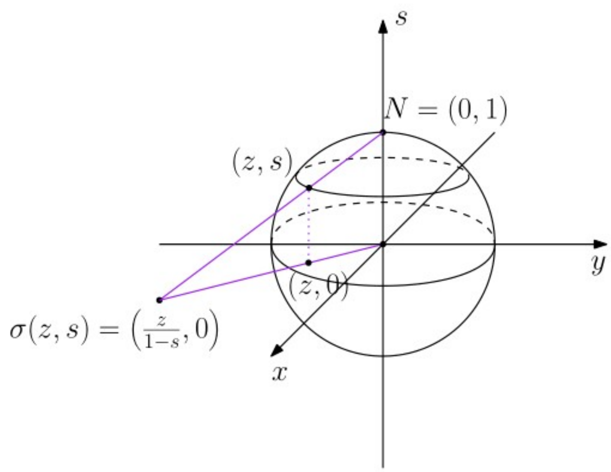

Figure 1: The Riemann sphere together with the image of a point $(z, s) \in S^2 \setminus \{N\}$ in $\mathbb{C}$

With a little algebra, we have the stereographic projection (now including $N = (0, 1)$)

$$\sigma : S^2 \to \overline{\mathbb{C}}, \qquad (z, s) \mapsto \begin{cases} \frac{z}{1-s}, & \text{for } (z, s) \neq N, \\ \infty, & \text{for } (z, s) = N, \end{cases}$$

and its inverse

$$\tau : \overline{\mathbb{C}} \to S^2 \qquad z \mapsto \begin{cases} \left(\frac{2z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right), & \text{for } z \in \mathbb{C}, \\ N, & \text{for } z = \infty. \end{cases}$$

The sets S^2 with the induced topology from $\mathbb{R}^3$ and $\overline{\mathbb{C}}$ with the topology declared in Definition 2.20 are in fact **homeomorphic**, with homeomorphism given by σ.

10/3 Lecture

3 Holomorphic Functions

3.1 Differentiation of Complex Functions

In this chapter we introduce the central concept of this lecture, which is that of a **holomorphic** function. We will see later, that the term **analytic** function (which in real analysis designates functions that are locally given by a convergent power series) can in fact be used synonymously.

Definition 3.1. Let $A \subseteq \mathbb{C}$ be an open set. A function $f : A \to \mathbb{C}$ is said to be **differentiable (in the complex sense)** at $z_0 \in A$, if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists in $\mathbb{C}$. A different notation is $\frac{d}{dz} f(z_0) = f'(z_0)$. We call $f'(z_0) \in \mathbb{C}$ the **complex derivative** of f at z_0. If f is differentiable for all $z_0 \in A$, we say that f is **holomorphic** (or **analytic**) on A (or simply holomorphic / analytic). We say that f is **holomorphic** at z_0, if there is an open r-disk $D(z_0, r) \subseteq A$ such that $f|_{D(z_0,r)}$ is holomorphic. A function that is defined on $\mathbb{C}$ and is holomorphic is called **entire**.

Example 3.2. (1) Constant functions $f \equiv c \in \mathbb{C}$ are entire with $f'(z_0) = 0$ for every $z_0 \in \mathbb{C}$.

(2) The function $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto z^n$ for integer $n \geq 1$ is entire. Indeed, using the binomial theorem, we see that for any $z_0 \in \mathbb{C}$,

$$\lim_{h \to 0} \frac{(z_0 + h)^n - z_0^n}{h} = \lim_{h \to 0} \frac{1}{h} \left(\sum_{k=0}^{n} \binom{n}{k} z_0^{n-k} h^k - z_0^n \right) = \lim_{h \to 0} \left(n z_0^{n-1} + h \sum_{k=2}^{n} \binom{n}{k} z_0^{n-k} h^{k-2} \right) = n z_0^{n-1}.$$

(3) The function $\mathrm{Re} : \mathbb{C} \to \mathbb{C}$, $z \mapsto \mathrm{Re}(z)$ is not differentiable (in the complex sense) in any point $z_0 \in \mathbb{C}$. Indeed, consider for $z_0 \in \mathbb{C}$, $h \neq 0$, that

$$\frac{\mathrm{Re}(z_0 + h) - \mathrm{Re}(z_0)}{h} = \frac{1}{h} \mathrm{Re}(h) = \begin{cases} 1, & \text{for } h \in \mathbb{R}, \\ 0, & \text{for } h \in \mathbb{R}i. \end{cases}$$

(4) In the same manner, $z \mapsto \mathrm{Im}(z)$ and $z \mapsto \overline{z}$ are not differentiable (in the complex sense) in any point $z_0 \in \mathbb{C}$. The function $z \mapsto |z|$ is only differentiable (in the complex sense) at $z_0 = 0$ (it is not holomorphic at $z_0 = 0$).

Lemma 3.3. Let $A \subseteq \mathbb{C}$ be open, $z_0 \in A$ and $f : A \to \mathbb{C}$ a function. The following are equivalent:

(1) f is differentiable in the complex sense in z_0 and $f'(z_0) = w$.

(2) There exists a function $\phi : A \to \mathbb{C}$, continuous at z_0 with

$$f(z) = f(z_0) + \phi(z)(z - z_0), \quad \text{and } \phi(z_0) = w.$$

Proof. Assume that f is differentiable at z_0 with $w = f'(z_0)$. We set

$$\phi(z) = \begin{cases} \frac{f(z) - f(z_0)}{z - z_0}, & \text{for } z \in A \setminus \{z_0\}, \\ w, & \text{for } z = z_0. \end{cases}$$

Since $\lim_{z \to z_0} \phi(z) = f'(z_0) = w = \phi(z_0)$, the function ϕ is continuous. On the other hand, suppose that a function ϕ with the desired properties exists, then

$$\frac{f(z) - f(z_0)}{z - z_0} = \phi(z),$$

and taking $z \to z_0$ on both sides yields by the continuity of ϕ that $f'(z_0)$ exists and equals $\phi(z_0) = w$. $\square$

Proposition 3.4. Suppose that $A \subseteq \mathbb{C}$ is open and f is differentiable (in the complex sense) at $z_0 \in A$. Then f is continuous at z_0.

Proof. By Lemma 3.3, we can write

$$\lim_{z \to z_0} f(z) = f(z_0) + \lim_{z \to z_0} (\phi(z)(z - z_0)) = f(z_0) + \lim_{z \to z_0} \phi(z) \cdot \lim_{z \to z_0} (z - z_0) = f(z_0),$$

since $\lim_{z \to z_0} \phi(z) = f'(z_0)$ and $\lim_{z \to z_0} (z - z_0) = 0$. $\square$

Proposition 3.5. Suppose that $A \subseteq \mathbb{C}$ is open and f, g are differentiable (in the complex sense) at $z_0 \in A$. Then the following statements hold:

(1) $f + g$ is differentiable at z_0 and has derivative $(f+g)'(z_0) = f'(z_0) + g'(z_0)$.

(2) $f \cdot g$ is differentiable at z_0 and has derivative $(f \cdot g)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$.

(3) If $g(z_0) \neq 0$, then $\frac{f}{g}$ (which can be defined on $D(z_0, \delta)$ for some $\delta > 0$) is differentiable at z_0 and has derivative

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}.$$

(4) If $h : B \to \mathbb{C}$ is a function defined on an open set $B \supseteq f(A)$, that is differentiable in $f(z_0)$, the composition $h \circ f$ is differentiable at z_0 and has derivative $(h \circ f)'(z_0) = h'(f(z_0))f'(z_0)$.

Proof. (1) Trivial.

(2) Same as in the real case.

(3) Same as in the real case.

(4) By Lemma 3.3, there is a function $\phi : B \to \mathbb{C}$, continuous at $f(z_0)$ with

$$h(w) = h(f(z_0)) + \phi(w)(w - f(z_0)), \quad \text{and } \phi(f(z_0)) = h'(f(z_0)).$$

By choosing $w = f(z)$ (for $z \neq z_0$), we have

$$\frac{h(f(z)) - h(f(z_0))}{z - z_0} = \phi(f(z))\frac{f(z) - f(z_0)}{z - z_0}.$$

Again by Lemma 3.3, there is a function $\psi : A \to \mathbb{C}$, continuous at z_0 with

$$f(z) - f(z_0) = \psi(z)(z - z_0), \quad \text{and } \psi(z_0) = f'(z_0).$$

We combine these two equations and find that

$$\lim_{z \to z_0} \frac{h(f(z)) - h(f(z_0))}{z - z_0} = \lim_{z \to z_0} (\phi(f(z))\psi(z)) = \phi(f(z_0))\psi(z_0) = h'(f(z_0))f'(z_0).$$

□

Example 3.6. Polynomial functions $P : \mathbb{C} \to \mathbb{C}$, $z \mapsto \sum_{v=0}^{n} a_v z^v$ are entire and have derivative

$$P'(z) = \sum_{v=1}^{n} v a_v z^{v-1}.$$

Rational functions $f : \mathbb{C} \backslash \{z_1, \cdots, z_n\}$ with $f(z) = \frac{P(z)}{Q(z)}$, where $z_1, \cdots, z_n$ are the zeros of Q are holomorphic, and their derivative can be calculated by the quotient rule.

3.2 The Cauchy-Riemann Equations

We introduced $\mathbb{C}$ as $\mathbb{R}^2$ with an additional multiplication structure. From real analysis, one has a notion of **(total) differentiablity** of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$. Naturally, the question arises that how these two notions are related. We already saw that the function $\mathrm{Re} : \mathbb{C} \to \mathbb{C}$, $z \mapsto \mathrm{Re}(z)$ is not differentiable in the complex sense, even though the corresponding function $(x, y) \mapsto (x, 0)$ viewed as a map from $\mathbb{R}^2 \to \mathbb{R}^2$ is totally differentiable. We shall see that the complex structure forces certain relations between the (real) partial derivatives, if a function is to be differentiable in the complex sense.

Let $A \subseteq \mathbb{C}$ be an open set and $f : A \to \mathbb{C}$. Recall that $z = x + yi = (x, y)$. We define the functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ as

$$u(x, y) = \mathrm{Re}(f(x, y)), \qquad v(x, y) = \mathrm{Im}(f(x, y)),$$

meaning that $f(x,y) = u(x,y) + iv(x,y)$. To stress the viewpoint as a map from a subset of $\mathbb{R}^2$ into $\mathbb{R}^2$, we may use the column vector notation

$$f : A \to \mathbb{R}^2, \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}.$$

We recall the notion of **Jacobi matrix** or **Jacobian** $Df(x,y)$, given by the partial derivatives

$$Df(x,y) = \begin{pmatrix} \frac{\partial u(x,y)}{\partial x} & \frac{\partial u(x,y)}{\partial y} \\ \frac{\partial v(x,y)}{\partial x} & \frac{\partial v(x,y)}{\partial y} \end{pmatrix}.$$

We also recall that the function $f : A \to \mathbb{R}^2$ is called **totally differentiable** at $(x_0, y_0) \in A$, if there is a matrix $M \in \mathrm{Mat}(2 \times 2, \mathbb{R})$ such that for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\frac{\left\| f(x,y) - f(x_0,y_0) - M \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \right\|}{\left\| \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \right\|} < \epsilon, \quad \text{for } \left\| \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \right\| < \delta.$$

Here $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^2$. In this case, the functions u and v admit partial derivatives in (x_0, y_0) and M is necessarily given by the Jacobian

$$M = Df(x_0, y_0).$$

We can now formulate the fundamental connection between real and complex differentiablity.

Theorem 3.7. Let $A \subseteq \mathbb{C}$ be open, $z_0 \in A$, and $f : A \to \mathbb{C}$ a function with decomposition $f(x,y) = u(x,y) + iv(x,y)$ and $u(x,y), v(x,y) \in \mathbb{R}$. The following are equivalent:

(1) f is differentiable at z_0 in the complex sense.

(2) f is totally differentiable at (x_0, y_0) in the real sense and at (x_0, y_0), the partial derivatives fulfill the **Cauchy-Riemann equations**

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \tag{1}$$
$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0). \tag{2}$$

Moreover, the complex derivative $f'(z_0)$ can be expressed as follows:

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) \tag{3}$$
$$if'(z_0) = i\frac{\partial v}{\partial y}(x_0, y_0) + \frac{\partial u}{\partial y}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0). \tag{4}$$

Proof. Let us first assume that f is differentiable in the complex sense at z_0 with $f'(z_0) = w$. Then, by definition,

$$\lim_{h \to 0} \left| \frac{f(z_0 + h) - f(z_0)}{h} - w \right| = \lim_{h \to 0} \left| \frac{f(z_0 + h) - f(z_0) - wh}{h} \right| = 0,$$

which implies that

$$\lim_{h \to 0} \frac{|f(z_0 + h) - f(z_0) - wh|}{|h|} = 0.$$

We write $w = a + bi$ with $a, b \in \mathbb{R}$ and $h = k + li$ with $k, l \in \mathbb{R}$, so $wh = ak - bl + (al + bk)i$, thus we can rewrite the previous equation as

$$\lim_{(k,l) \to (0,0)} \frac{\left\| f(x_0 + k, y_0 + l) - f(x_0, y_0) - \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} k \\ l \end{pmatrix} \right\|}{\left\| \begin{pmatrix} k \\ l \end{pmatrix} \right\|} = 0.$$

The above equation means that f is totally differentiable in $(x_0, y_0) \in \mathbb{R}^2$ and the corresponding Jacobian is

$$Df(x_0, y_0) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Thus, we must have

$$\begin{aligned} \frac{\partial u}{\partial x}(x_0, y_0) &= a = \frac{\partial v}{\partial y}(x_0, y_0), \\ \frac{\partial u}{\partial y}(x_0, y_0) &= -b = -\frac{\partial v}{\partial x}(x_0, y_0). \end{aligned}$$

Conversely, assume that f is totally differentiable in (x_0, y_0) in the real sense, with partial derivatives of u and v fulfilling the Cauchy-Riemann equations (1–2). By defining a and b as above, and then $w = a + bi$ and $h = k + li$, we can recover all the previous steps in the reverse direction, and thus f is differentiable in the complex sense at z_0. Finally, we note that since $w = f'(z_0) = a + bi$, the forms (3–4) directly follow. $\square$

Corollary 3.8. Let $A \subseteq \mathbb{C}$ be open and $f : A \to \mathbb{C}$ a function.

(1) If all partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$ exist, are continuous on A, and fulfill the Cauchy Riemann equations (1–2), then f is holomorphic.

(2) Assume additionally that A is connected. If f is holomorphic and attains only real values, it must be constant.

Proof. (1) Note that this means f is totally differentiable in the real sense, then Theorem 3.7 would apply.

(2) This is trivial since $v \equiv 0$.

$\square$

Remark 3.9. In fact, one can observe that both u and v are harmonic functions: We recall that a function $g : A \to \mathbb{R}$ defined on an open subset $A \subseteq \mathbb{R}^2$, is called **harmonic function** if its is twice differentiable and

$$\Delta g(x, y) = \frac{\partial^2 g}{\partial x^2}(x, y) + \frac{\partial^2 g}{\partial y^2}(x, y) = 0, \quad \text{for } (x, y) \in A.$$

This can be easily done by applying Schwarz' theorem on the interchangeability of the order of partial derivatives for twice continuously differentiable functions.

Proposition 3.10. The exponential function $\exp : \mathbb{C} \to \mathbb{C}$, $z \mapsto \exp(z)$ is entire and fulfills

$$\exp'(z) = \exp(z), \quad \forall z \in \mathbb{C}.$$

Proof. By definition $f(z) = f(x, y) = \exp(x)(\cos(x) + i\sin(y))$, so we have $u(x, y) = \exp(x)\cos(x)$ and $v(x, y) = \exp(x)\sin(y)$. These functions are smooth (infinitely differentiable in the sense of real variables), and they fulfill

$$\begin{aligned} \frac{\partial u}{\partial x}(x, y) &= \exp(x)\cos(y) = \frac{\partial v}{\partial y}(x, y), \\ \frac{\partial u}{\partial y}(x, y) &= -\exp(x)\sin(y) = -\frac{\partial v}{\partial x}(x, y). \end{aligned}$$

As a result, exp is entire by Corollary 3.8. Then (3) gives

$$\exp'(z) = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y) = \exp(x)(\cos(y) + i\sin(y)) = \exp(z).$$

$\square$

Corollary 3.11. The trigonometric functions $\sin : \mathbb{C} \to \mathbb{C}$ and $\cos : \mathbb{C} \to \mathbb{C}$ are entire and fulfill

$$\sin'(z) = \cos(z), \quad \cos'(z) = -\sin(z), \quad \forall z \in \mathbb{C}.$$

Proof. The proof is trivial by direct applications of complex derivative properties. $\square$

Corollary 3.12. Let $\mathbb{C}_- = \mathbb{C} \setminus \{x + yi \in \mathbb{C};\ x \leq 0,\ y = 0\}$. Then, the restriction of Log is holomorphic with

$$\mathrm{Log}'(z) = \frac{1}{z}, \quad \forall z \in \mathbb{C}_-.$$

Proof. The proof uses either the inverse function theorem, or the Cauchy-Riemann equations in polar coordinates, which is left as an exercise. $\square$

10/11 Lecture

4 Contour Integrals and Cauchy's Theorem

4.1 Contour Integrals

Definition 4.1. Let $I = [a, b] \subseteq \mathbb{R}$ with $a < b$ be a closed interval and $\gamma : I \to \mathbb{C}$ a map. We denote $\gamma = \gamma_1 + \gamma_2 i$ with $\gamma_1(t), \gamma_2(t) \in \mathbb{R}$ for every $t \in [a, b]$. We say that γ is **continuous** / **differentiable**[2] / **continuously differentiable** if this holds for both γ_1 and γ_2. If γ is differentiable at $t \in I$, we set

$$\gamma'(t) = \gamma_1'(t) + i\gamma_2'(t).$$

We say that a continuous map γ is **piecewise continuously differentiable** if there exists a partition $a = a_0 < a_1 < \cdots < a_n = b$ of I such that $\gamma|_{[a_{j-1}, a_j]}$ is continuously differentiable for every $1 \leq j \leq n$.

By convention, the term **curve** will always mean a continuous, piecewise continuously differentiable map.

Lemma 4.2. (1) Sums and products of differentiable functions $\gamma, \delta : I \to \mathbb{C}$ are differentiable, and for $t \in [a, b]$, we have

$$\begin{aligned}(\gamma + \delta)'(t) &= \gamma'(t) + \delta'(t), \\ (\gamma \cdot \delta)'(t) &= \gamma'(t)\delta(t) + \gamma(t)\delta'(t).\end{aligned}$$

(2) Let $A \subseteq \mathbb{C}$ be open and $\gamma : I \to \mathbb{C}$ differentiable with $\gamma(I) \subseteq A$. For $f : A \to \mathbb{C}$ holomorphic, the function

$$\delta : I \to \mathbb{C}, \quad t \mapsto f(\gamma(t))$$

is differentiable and has derivative in $t \in I$

$$\delta'(t) = f'(\gamma(t))\gamma'(t).$$

Proof. (1) Trivial.

(2) Write $f = u + vi$ and $\gamma = \gamma_1 + \gamma_2 i$, then

$$\delta(t) = u(\gamma_1(t), \gamma_2(t)) + iv(\gamma_1(t), \gamma_2(t)).$$

This is differentiable in the real sense (as a map from $I \to \mathbb{R}^2 = \mathbb{C}$). By the chain rule from real analysis, we thus find

$$\begin{aligned}\delta'(t) &= \frac{\partial u}{\partial t}(\gamma_1(t), \gamma_2(t)) + i\frac{\partial v}{\partial t}(\gamma_1(t), \gamma_2(t)) \\ &\quad \text{by Cauchy-Riemann equations (3)} \\ &= \frac{\partial u}{\partial x}(\gamma_1(t), \gamma_2(t))\gamma_1'(t) + \frac{\partial u}{\partial y}(\gamma_1(t), \gamma_2(t))\gamma_2'(t) + i\left(\frac{\partial v}{\partial x}(\gamma_1(t), \gamma_2(t))\gamma_1'(t) + \frac{\partial v}{\partial y}(\gamma_1(t), \gamma_2(t))\gamma_2'(t)\right) \\ &= \left(\frac{\partial u}{\partial x}(\gamma_1(t), \gamma_2(t)) + i\frac{\partial v}{\partial x}(\gamma_1(t), \gamma_2(t))\right)(\gamma_1'(t) + i\gamma_2'(t)) = f'(\gamma(t))\gamma'(t). \\ &\quad \text{by Cauchy-Riemann equations (1–2) and (3)}\end{aligned}$$

□

We now define two types of integrals: The integral of a curve $\gamma : I \to \mathbb{C}$ and the integral of a continuous complex function **along** a curve.

Definition 4.3. Let $I = [a, b] \subseteq \mathbb{R}$ with $a < b$ and $\gamma : I \to \mathbb{C}$ continuous with $\gamma = \gamma_1 + \gamma_2 i$ (so γ_1, γ_2 are continuous). We define the integrals of γ as

$$\int_a^b \gamma(t)dt = \int_a^b \gamma_1(t)dt + i\int_a^b \gamma_2(t)dt, \quad \text{and} \quad \int_b^a \gamma(t)dt = -\int_a^b \gamma(t)dt.$$

[2]Recall: $f : [a, b] \to \mathbb{R}$ is differentiable in $[a, b]$ if it is differentiable in (a, b), and the limits $\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$ and $\lim_{x \to b^+} \frac{f(x) - f(b)}{x - b}$ exist.

In other words, the integral is obtianed by integrating componentwise the vector-valued function $\gamma : i \to \mathbb{R}^2$. The continuity of γ ensures the continuity of γ_1 and γ_2 on the bounded and closed (hence compact) set $I = [a, b]$, so all integrals are well-defined.

Lemma 4.4. Let $I = [a, b] \subseteq \mathbb{R}$ with $a < b$ and $\gamma, \delta : I \to \mathbb{C}$ continuous.

(1) For all $a < c < b$, we have

$$\int_a^b \gamma(t)dt = \int_a^c \gamma(t)dt + \int_c^b \gamma(t)dt.$$

(2) For all $\lambda \in \mathbb{C}$, we have

$$\int_a^b (\lambda\gamma(t) + \delta(t))dt = \lambda \int_a^b \gamma(t)dt + \int_a^b \delta(t)dt.$$

(3) The real and imaginary parts of the integral fulfill

$$\operatorname{Re}\left(\int_a^b \gamma(t)dt\right) = \int_a^b \operatorname{Re}(\gamma(t))dt, \quad \operatorname{Im}\left(\int_a^b \gamma(t)dt\right) = \int_a^b \operatorname{Im}(\gamma(t))dt.$$

(4) The map $t \mapsto \int_a^t \gamma(x)dx$ is differentiable on I and has the derivative

$$\frac{d}{dt}\left(\int_a^t \gamma(x)dx\right) = \gamma(t).$$

(5) If $\gamma : I \to \mathbb{C}$ is continuously differentiable, then

$$\int_a^b \gamma'(t)dt = \gamma(b) - \gamma(a).$$

(6) The modulus of the integral fulfills

$$\left|\int_a^b \gamma(t)dt\right| \leq \int_a^b |\gamma(t)|dt.$$

Proof. (1) Trivial.

(2) Trivial.

(3) Trivial.

(4) This follows from Fundamental Theorem of Calculus, Part I, if applied componentwise.

(5) This follows from Fundamental Theorem of Calculus, Part II, if applied componentwise.

(6)

□

Definition 4.5. Let $A \subseteq \mathbb{C}$ be open and $f : A \to \mathbb{C}$ continuous. Let $I = [a, b] \subseteq \mathbb{R}$ with $a < b$ and $\gamma : I \to \mathbb{C}$ a piecewise continuously differentiable curve with $\gamma(I) \subseteq A$. We define the **integral of f along γ** as

$$\int_\gamma f(z)dz = \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} f(\gamma(t))\gamma'(t)dt,$$

where $a = a_0 < a_1 < \cdots < a_n = b$ is a partition such that $\gamma|_{[a_{j-1}, a_j]}$ is continuously differentiable for every $1 \leq j \leq n$.

Remark 4.6. (1) Since $t \mapsto \gamma'(t)$ is continuous on $[a_{j-1}, a_j]$ and $t \mapsto f(\gamma(t))$ is continuous as composition of continuous maps, the product $t \mapsto f(\gamma(t))\gamma'(t)$ itself is a continuous map from $[a_{j-1}, a_j]$ to $\mathbb{C}$, and the integrals on the right-hand side are defined as before.

(2) It is easy to show that the definition does not depend on the choice of the partition: If $a = \tilde{a}_0 < \tilde{a}_1 < \cdots < \tilde{a}_m = b$ is another partition on $[a, b]$ such that $\gamma|_{[\tilde{a}_{j-1},\tilde{a}_j]}$ is continuously differentiable for every $1 \leq j \leq m$, we can choose a common refinement $a = \xi_0 < \cdots < \xi_k = b$ (such that every ξ_j corresponds to some a_l or $\tilde{a}_l$). Then this proof will follow trivially.

(3) If we write $f(z) = u(x, y) + iv(x, y)$ with $u, v \in \mathbb{R}$, we can calculate the complex contour integral as

$$\int_\gamma f(z)dz = \int_a^b (u(\gamma_1(t), \gamma_2(t))\gamma_1'(t) - v(\gamma_1(t), \gamma_2(t))\gamma_2'(t))dt + i\int_a^b (u(\gamma_1(t), \gamma_2(t))\gamma_2'(t) - v(\gamma_1(t), \gamma_2(t))\gamma_1'(t))dt.$$

Proposition 4.7. Let $A \subseteq \mathbb{C}$ be open, $f : A \to \mathbb{C}$ continuous. Let

$$I = [a, b] \subseteq \mathbb{R} \text{ with } a < b \text{ and } \gamma : I \to \mathbb{C} \text{ piecewise continuously differentiable with } \gamma(I) \subseteq A,$$
$$\tilde{I} = [\tilde{a}, \tilde{b}] \subseteq \mathbb{R} \text{ with } \tilde{a} < \tilde{b} \text{ and } \tilde{\gamma} : \tilde{I} \to \mathbb{C} \text{ piecewise continuously differentiable.}$$

Assume that there exists a continuously differentiable function $\tau : I \to \tilde{I}$ with $\tau'(t) > 0$ for every $t \in [a, b]$, fulfilling $\tau(a) = \tilde{a}$, $\tau(b) = \tilde{b}$, and $\gamma = \tilde{\gamma} \circ \tau$. Then it holds that $\tilde{\gamma}(\tilde{I}) = \gamma(I)$, and we have

$$\int_{\tilde{\gamma}} f(z)dz = \int_\gamma f(z)dz.$$

Proof. Since τ is continuously differentiable with positive derivative, it is an increasing and continuous function, so since $\tau(a) = \tilde{a}$ and $\tau(b) = \tilde{b}$, we van see that $\tau(I) = \tilde{I}$. By breaking up I into subintervals (if necessary), we assume without loss of generality that $\gamma \in C^1$. Using the chain rule, we see that

$$\gamma'(t) = \frac{d}{dt}\tilde{\gamma}(\tau(t)) = \tilde{\gamma}'(\tau(t))\tau'(t).$$

Using the substitution rule, we see that

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b f(\tilde{\gamma}(\tau(t)))\tilde{\gamma}'(\tau(t))\tau'(t)dt = \int_{\tilde{a}}^{\tilde{b}} f(\tilde{\gamma}(s))\tilde{\gamma}'(s)ds = \int_{\tilde{\gamma}} f(z)dz.$$

□

Definition 4.8. (1) Let $I = [a, b. \subseteq \mathbb{R}$ with $a < b$ and $\gamma : I \to \mathbb{C}$ a continuous curve. The map

$$-\gamma : I \to \mathbb{C}, \quad t \mapsto \gamma(a + b - t)$$

is called the **opposite curve** of γ.

(2) Let $I_1 = [a, b] \subseteq \mathbb{R}$, $I_2 = [b, c] \subseteq \mathbb{R}$ with $a < b < c$ and $\gamma_1 : I_1 \to \mathbb{C}$, $\gamma_2 : I_2 \to \mathbb{C}$ two continuous curves with $\gamma_1(b) = \gamma_2(b)$. Let $I = I_1 \cup I_2 = [a, c]$. The map

$$\gamma_1 + \gamma_2 : I \to \mathbb{C}, \quad t \mapsto \begin{cases} \gamma_1(t), & t \in [a, b], \\ \gamma_2(t), & t \in [b, c], \end{cases}$$

is called the **join** or **union** or **sum** of the curves γ_1 and γ_2 (also denoted $\gamma_1 * \gamma_2$)[3].

Proposition 4.9. Let $A \subseteq \mathbb{C}$ and $f, g : A \to \mathbb{C}$ continuous. Let $I = [a, b] \subseteq \mathbb{R}$ with $a < b$ and let $\gamma : I \to \mathbb{C}$ be a piecewise continuously differentiable curve with $\gamma(I) \subseteq A$.

(1) For every $\lambda \in \mathbb{C}$, we have

$$\int_\gamma (\lambda f(z) + g(z))dz = \lambda \int_\gamma f(z)dz + \int_\gamma g(z)dz.$$

[3] By slight abuse of notation, we also define the join of $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [c, d] \to \mathbb{C}$ if $\gamma_1(b) = \gamma_2(c)$ as the join of γ_1 and $\tilde{\gamma}_2 : \tilde{I}_2 := [b, b + d - c] \to \mathbb{C}$, where $\tilde{\gamma}_2(t) = \gamma_2(t + c - b)$ is a reparametrization of γ_2, *i.e.*, $\gamma_1 + \gamma_2 = \gamma_1 + \tilde{\gamma}_2$, which maps $I = I_1 \cup \tilde{I}_2 = [a, b + d - c]$ into $\mathbb{C}$.

(2) For the integral along $-\gamma$, it holds that

$$\int_{-\gamma} f(z)dz = -\int_{\gamma} f(z)dz.$$

(3) Suppose $I' = [b, c]$ with $b < c$ and $\delta : I' \to \mathbb{C}$ is another piecewise continuously differentiable curve with $\delta(I') \subseteq A$ and $\delta(b) = \gamma(b)$. Then

$$\int_{\gamma+\delta} f(z)dz = \int_{\gamma} f(z)dz + \int_{\delta} f(z)dz.$$

Proof. The proofs are trivial and will not be specified. (1) follows directly from the properties of curve integrals. (2) follows by manipulating the integrals by change of variables. (3) requires the definition of the join of two curves and that of the contour integrals. $\square$

Example 4.10. (1) Let $z_0 \in \mathbb{C}$ and consider the integral

$$\int_{\gamma} \frac{1}{z - z_0} dz,$$

where γ is the full circle with radius $R > 0$ centered at z_0, traversed in a counterclockwise sense. A simple parametrization of γ is

$$\gamma : [0, 2\pi] \to \mathbb{C}, \quad t \mapsto R\exp(it) + z_0.$$

We therefore calculate

$$\int_{\gamma} \frac{1}{z - z_0} dz = \int_0^{2\pi} \frac{1}{R\exp(it)} \cdot iR\exp(it)dt = 2\pi i.$$

(2) Let us compute

$$\int_{\gamma} z^3 dz,$$

where γ is the line segment between 0 and $1 + i$. A simple parametrization for γ is

$$\gamma : [0, 1] \to \mathbb{C}, \quad t \mapsto (1 + i)t.$$

We therefore calculate

$$\int_{\gamma} z^3 dz = \int_0^1 (1 + i)^3 t^3 \cdot (1 + i)dt = -4\int_0^1 t^3 dt = -1.$$

Definition 4.11. Let $I = [a, b] \subseteq \mathbb{R}$ with $a < b$ and γ be a piecewise continuously differentiable curve, such that $\gamma|_{[a_{j-1},a_j]}$ for $j = 1, \cdots, n$ are continuously differentiable where $a = a_0 < a_1 < \cdots < a_n = b$. Then the quantity

$$l(\gamma) = \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} |\gamma'(t)|dt$$

is called the **arc length** of γ.

Remark 4.12. (1) The arc length is well defined and finite. Indeed, since the function $\gamma_j : [a_{j-1}, a_j] \to \mathbb{C}$, $t \mapsto \gamma'(t)$ is continuous and the modulus function $|\cdot| : \mathbb{C} \to \mathbb{R}$, $z \mapsto |z|$ is continuous, their composition $t \mapsto |\gamma'(t)|$ is a continuous map from a compact interval into $\mathbb{R}$, so the claim follows.

(2) The arc length is invariant under reparametrization.

Definition 4.13. Let $A \subseteq \mathbb{C}$ be open and $f : A \to \mathbb{C}$ continuous. Let $I = [a, b] \subseteq \mathbb{R}$ with $a < b$ and $\gamma : I \to \mathbb{C}$ a piecewise continuously differentiable curve with $\gamma(I) \subseteq A$. We define the **integral of f with respect to the length of γ** as

$$\int_\gamma f(z)ds = \int_\gamma f(z)|dz| = \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} f(\gamma(t))|\gamma'(t)|dt,$$

where $a = a_0 < a_1 < \cdots < a_n = b$ is a partition such that $\gamma|_{[a_{j-1},a_j]}$ is continuously differentiable for every $1 \leq j \leq n$. This integral is again invariant under reparametrization, and note that by definition, we have

$$l(\gamma) = \int_\gamma ds.$$

Proposition 4.14. Let $A \subseteq \mathbb{C}$ be open and $f : A \to \mathbb{C}$ continuous. Let $I = [a, b] \subseteq \mathbb{R}$ with $a < b$ and $\gamma : I \to \mathbb{C}$ a piecewise continuously differentiable curve with $\gamma(I) \subseteq A$. Then, we have that

$$\left|\int_\gamma f(z)dz\right| \leq \int_\gamma |f(z)|ds.$$

In particular, if $\sup_{z\in\gamma(I)} |f(z)| \leq M \in [0, \infty)$, then

$$\left|\int_\gamma f(z)dz\right| \leq Ml(\gamma).$$

Proof. As usual consider a partition $a = a_0 < a_1 < \cdots < a_n = b$ such that $\gamma|_{[a_{j-1},a_j]}$, $j = 1, \cdots, n$ are all continuously differentiable. We use the definition of the contour integral and apply the bound of modulus to obtain

$$\left|\int_\gamma f(z)dz\right| = \left|\sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} f(\gamma(t))\gamma'(t)dt\right| \leq \sum_{j=1}^{n} \left|\int_{a_{j-1}}^{a_j} f(\gamma(t))\gamma'(t)dt\right| \leq \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} |f(\gamma(t))||\gamma'(t)|dt = \int_\gamma |f(z)|ds.$$

The second part easily follows by compactness and the given supremum. ☐

Theorem 4.15 (Fundamental Theorem of Calculus for Contour Integrals)**.** Let $A \subseteq \mathbb{C}$ be open, $I = [a, b] \subseteq \mathbb{R}$ with $a < b$ and $\gamma : I \to \mathbb{C}$ be a piecewise continuously differentiable curve with $\gamma(I) \subseteq A$. Assume that $f : A \to \mathbb{C}$ is continuous and has a primitive F (this means that $F : A \to \mathbb{C}$ is holomorphic and fulfills $F'(z) = f(z)$ for all $z \in A$). Then,

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

In particular, if γ is a closed curve (*i.e.*, $\gamma(a) = \gamma(b)$), then

$$\int_\gamma f(z)dz = 0.$$

Proof. As usual consider a partition $a = a_0 < a_1 < \cdots < a_n = b$ such that $\gamma|_{[a_{j-1},a_j]}$, $j = 1, \cdots, n$ are all continuously differentiable. Then

$$\begin{aligned}\int_\gamma f(z)dz = \sum_{j=1}^{n} \int_{\gamma_j} f(z)dz = \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} f(\gamma(t))\gamma'(t)dt &= \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} F'(\gamma(t))\gamma'(t)dt = \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} \frac{d}{dt}F(\gamma(t))dt \\ &= \sum_{j=1}^{n}(F(\gamma(a_j)) - F(\gamma(a_{j-1}))) = F(\gamma(b)) - F(\gamma(a)).\end{aligned}$$

The second part is then immediate. ☐

Corollary 4.16. Let $A \subseteq \mathbb{C}$ be a domain and $f : A \to \mathbb{C}$ a holomorphic function with $f'(z) = 0$ for every $z \in A$. Then f is constant.

Proof. Fix $z_0 \in A$ and let $z \in A$ be another point, then there exists a piecewise continuously differentiable curve γ connecting z_0 to z. Since f is a primitive of f', we then have

$$f(z) - f(z_0) = \int_\gamma f'(w)dw = 0,$$

so f is constant as desired. ☐

10/17 Lecture

(Continued)

We have seen that the existence of a primitive allows the calculation of complex contour integrals in a very easy way, and the value of such an integral is manifestly independent of the geometry of the path γ, and only depends on the endpoints $\gamma(a)$ and $\gamma(b)$. The next result shows that the path-independence of complex contour integrals is in fact equivalent to the existence of a primitive.

Theorem 4.17. Suppose $A \subseteq \mathbb{C}$ is a domain and $f : A \to \mathbb{C}$ is continuous. Then the following are equivalent:

(1) Integrals over f are path independent, *i.e.*, if $z_0, z_1 \in A$ and $\gamma_0 : I_0 = [a_0, b_0] \to \mathbb{C}$ and $\gamma_1 : I_1 = [a_1, b_1] \to \mathbb{C}$ (with $a_j < b_j$, $j = 0, 1$) are piecewise continuously differentiable curves with $\gamma_0(I_0), \gamma_1(I_1) \subseteq A$ fulfilling $\gamma_0(a_0) = \gamma_1(a_1) = z_0$ and $\gamma_0(b_0) = \gamma_1(b_1) = z_1$, then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

(2) Integrals over f along closed curves are zero, *i.e.*, if $\gamma : I = [a, b] \to \mathbb{C}$ (with $a < b$) is a piecewise continuously differentiable curve with $\gamma(I) \subseteq A$ and $\gamma(a) = \gamma(b)$, then

$$\int_{\gamma} f(z)dz = 0.$$

(3) There is a primitive of f on A, *i.e.*, there exists $F : A \to \mathbb{C}$ holomorphic with

$$F'(z) = f(z), \quad \text{for } z \in A.$$

Proof. We first prove the equivalence of (1) and (2). Let $\gamma : I \to \mathbb{C}$ be a piecewise continuously differentiable curve with $\gamma(I) \subseteq A$. We assume that γ is not constant, since otherwise $\int_\gamma f(z)dz = 0$ automatically. Then there exist $a < c < b$ and $\gamma(c) \neq \gamma(a) = \gamma(b)$. Now consider

$$\gamma_1 : [a, c] \to \mathbb{C}, \quad t \mapsto \gamma(t), \qquad \gamma_2 : [c, b] \to \mathbb{C}, \quad t \mapsto \gamma(t).$$

It follows that $\gamma = \gamma_1 + \gamma_2$, and therefore we have that

$$\int_{\gamma} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz - \int_{-\gamma_2} f(z)dz = 0,$$

since $\gamma_1(a) = (-\gamma_2)(c)$ and $\gamma_1(c) = (-\gamma_2)(b)$. This shows that (1) implies (2). Now assume that $\gamma_0 : I_0 \to \mathbb{C}$ and $\gamma_1 : I_1 \to \mathbb{C}$ are two piecewise continuously differentiable curves with $\gamma_0(I_0), \gamma_1(I_1) \subseteq A$ and $\gamma_0(a_0) = \gamma_1(a_1)$, $\gamma_0(b_0) = \gamma_1(b_1)$. Then $\gamma_0 + (-\gamma_1)$ is a piecewise continuously differentiable closed curve, so that

$$\int_{\gamma_0} f(z)dz - \int_{\gamma_1} f(z)dz = \int_{\gamma_0} f(z)dz + \int_{-\gamma_1} f(z)dz = \int_{\gamma_0 + (-\gamma_1)} f(z)dz = 0,$$

which proves that (2) implies (1) as well. Now it suffices to prove the equivalence of (1) and (3). In fact, Fundamental Theorem of Calculus for Contour Integrals has already shown that (3) implies (2), and thus implies (1), so we are left with showing (1) implies (3). Let $f : A \to \mathbb{C}$ be continuous and $z_0 \in A$. Assuming that (1) holds, we define

$$F : A \to \mathbb{C}, \quad z \mapsto \int_{\gamma_z} f(w)dw,$$

where $\gamma_z : [a, b] \to \mathbb{C}$ with $a < b$ is any piecewise continuously differentiable curve with $\gamma_z(a) = z_0$ and $\gamma_z(b) = z$. This function is well defined, since

- domains are path connected, and for $z_0, z \in A$, there is always a piecewise continuously differentiable path connecting z_0 to z;
- the integral is finite by the standard bound for any choice of the curve γ_z, since piecewise continuously differentiable curves have finite length;

- the value of the integral is independent of the path γ_z we use by the assumption.

We now need to show that F is holomorphic with $F' = f$ on A. Fix $\epsilon > 0$, then for any $z \in A$, take $\delta > 0$ small enough such that $|f(z) - f(w)| < \epsilon$ for every $w \in D(z, \delta)$ where also $D(z, \delta) \subseteq A$ (since A is open and f is continuous at z). We then let $\gamma_w = \gamma_z + \gamma_{[z,w]}$, where

$$\gamma_{[z,w]} : [0,1] \to \mathbb{C}, \quad t \mapsto tw + (1-t)z.$$

Note that $\gamma_{[z,w]}([0,1]) \subseteq D(z,\delta)$ and γ_w is a piecewise continuously differentiable curve connecting z_0 and w with image in A. For $w \in \dot{D}(z,\delta)$, we therefore have

$$F(w) - F(z) = \int_{\gamma_z + \gamma_{[z,w]}} f(u)du - \int_{\gamma_z} f(u)du = \int_{\gamma_{[z,w]}} f(u)du = \int_0^1 f(tw + (1-t)z) \cdot (w-z)dt.$$

Hence, we can obtain that

$$\left|\frac{F(w) - F(z)}{w - z} - f(z)\right| = \left|\int_0^1 f(tw + (1-t)z)dt - f(z)\right| \leq \int_0^1 |f(tw + (1-t)z) - f(z)|\, dt < \epsilon,$$

implying that $F'(z) = f(z)$ for every $z \in A$ by the arbitrariness of ϵ. □

4.2 Cauchy's Integral Theorem

We have seen that the existence of a primitive of a continuous function f is equivalent to integrals of f along closed curves being zero. On $\mathbb{C}$, a function that is merely continuous need not have a primitive, for instance it holds that

$$\int_{\partial D(0,1)} \overline{z} dz = 2\pi i \neq 0.$$

As we shall see, the situation is rather different if we assume f to be holomorphic rather than just continuous. Hence we aim to answer the following question: **Are there any domains on which holomorphic functions always have a primitive?**

Definition 4.18. For $z_1, z_2, z_3 \in \mathbb{C}$, we define the **triangle spanned by z_1, z_2, and z_3** by

$$\triangle(z_1, z_2, z_3) = \left\{\sum_{i=1}^{3} \lambda_i z_i;\ \lambda_i \geq 0,\ \forall i \in \{1,2,3\},\ \sum_{i=1}^{3} \lambda_i = 1\right\}.$$

We also define the piecewise continuously differentiable curve

$$\gamma_{\partial\triangle(z_1,z_2,z_3)} : [0,3] \to \mathbb{C}, \quad t \mapsto \begin{cases} z_1 + t(z_2 - z_1), & \text{for } t \in [0,1], \\ z_2 + (t-1)(z_3 - z_2), & \text{for } t \in [1,2], \\ z_3 + (t-2)(z_1 - z_3), & \text{for } t \in [2,3]. \end{cases}$$

We notice that the image of the curve $\gamma_{\partial\triangle(z_1,z_2,z_3)}$ forms the boundary of the triangle $\triangle(z_1,z_2,z_3)$, *i.e.*, it always holds that $\gamma_{\partial\triangle(z_1,z_2,z_3)}([0,3]) = \partial\triangle(z_1,z_2,z_3)$.

Lemma 4.19 (Lemma of Goursat)**.** Let $A \subseteq \mathbb{C}$ be open and $f : A \to \mathbb{C}$ a holomorphic function. Assume that $z_1, z_2, z_3 \in A$ such that $\triangle(z_1,z_2,z_3) \subseteq A$. Then

$$\int_{\gamma_{\partial\triangle(z_1,z_2,z_3)}} f(z)dz = 0.$$

Remark 4.20. (1) We stress that the entire closed triangle $\triangle(z_1,z_2,z_3)$ must be contained in A, and not only its boundary. For instance, $\triangle(-1-i, 1-i, i)$ is not contained in $\mathbb{C}\setminus\{0\}$.

(2) Goursat's original proof uses rectangles rather than triangles. The prood using triangles is due to Pringsheim.

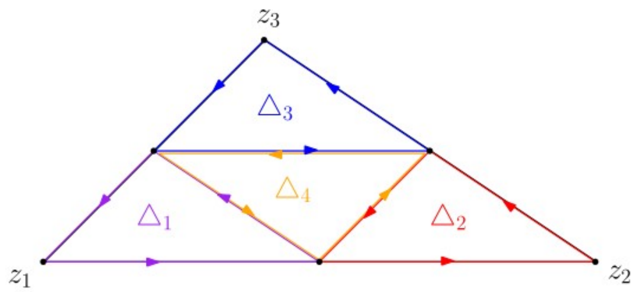

Figure 2: The partition of the triangle $\triangle_1, \triangle_2, \triangle_3, \triangle_4 \subseteq \triangle$

Proof of Lemma 4.19. We assume that none of z_1, z_2, and z_3 coincide, and none of the points is on the line connecting the other two, since otherwise the result is trivial. We also assume that $\triangle(z_1, z_2, z_3)$ is traversed in a counterclockwise direction, since otherwise we can consider $-\gamma_{\delta\triangle(z_1,z_2,z_3)}$, which will lead to a same result. We abbreviate $\triangle(z_1, z_2, z_3)$ by $\triangle$ and $\gamma_{\partial\triangle(z_1,z_2,z_3)}$ by $\gamma_{\partial\triangle}$. We then join the centers of the segments between each of the three points and subsequently obtain four triangles $\triangle_1, \triangle_2, \triangle_3, \triangle_4 \subseteq \triangle$, as is shown in Figure 2. All boundaries are traversed in a counterclockwise direction, and the respective piecewise continuously differentiable curves parametrizing the boundaries are denoted by $\gamma_{\partial\triangle_1}$, $\gamma_{\partial\triangle_2}$, $\gamma_{\partial\triangle_3}$, and $\gamma_{\partial\triangle_4}$. Then we have that

$$\int_{\gamma_{\partial\triangle}} f(z)dz = \int_{\gamma_{\partial\triangle_1}} f(z)dz + \int_{\gamma_{\partial\triangle_2}} f(z)dz + \int_{\gamma_{\partial\triangle_3}} f(z)dz + \int_{\gamma_{\partial\triangle_4}} f(z)dz,$$

since the parts of the integrals corresponding to integrals in opposite directions along the edges of $\triangle_4$ are cancelled out. Using the triangle inequality, we deduce that

$$\left|\int_{\gamma_{\partial\triangle}} f(z)dz\right| = \left|\int_{\gamma_{\partial\triangle_1}} f(z)dz\right| + \left|\int_{\gamma_{\partial\triangle_2}} f(z)dz\right| + \left|\int_{\gamma_{\partial\triangle_3}} f(z)dz\right| + \left|\int_{\gamma_{\partial\triangle_4}} f(z)dz\right|.$$

Now, let $\triangle^{(1)} \in \{\triangle_1, \triangle_2, \triangle_3, \triangle_4\}$ be the triangle (with piecewise continuously differentiable boundary curve $\gamma_{\partial\triangle^{(1)}}$ traversed in a counterclockwise direction) such that

$$\left|\int_{\gamma_{\partial\triangle^{(1)}}} f(z)dz\right| = \max_{j=1,2,3,4}\left|\int_{\gamma_{\partial\triangle_j}} f(z)dz\right|, \quad \text{choosing the smallest } j \text{ if argmax is not unique.}$$

Then we have that

$$\left|\int_{\gamma_{\partial\triangle}} f(z)dz\right| \le 4 \cdot \left|\int_{\gamma_{\partial\triangle^{(1)}}} f(z)dz\right|.$$

Iterating such a construction, we may find a sequence of closed triangles $\triangle =: \triangle^{(0)} \supseteq \triangle^{(1)} \supseteq \triangle^{(2)} \supseteq \cdots$, and we have in general that

$$\triangle^{(n+1)} \subseteq \triangle^{(n)}, \quad \left|\int_{\gamma_{\partial\triangle}} f(z)dz\right| \le 4^n \cdot \left|\int_{\gamma_{\partial\triangle^{(n)}}} f(z)dz\right|, \quad \forall n \in \mathbb{N}.$$

Since $\left(\triangle^{(n)}\right)_{n=0}^{\infty}$ are all closed and bounded, they are compact, so there exists a point $z_0 \in \bigcap_{n=0}^{\infty} \triangle^{(n)}$ which is proven to be nonempty. Moreover, for a given $\delta > 0$, there exists $N \in \mathbb{N}$, such that

$$\triangle^{(n)} \subseteq D(z_0, \delta), \quad \forall n \ge N,$$

since $z_0 \in \triangle^{(n)}$, and the diameter $d^{(n)}$ of $\triangle^{(n)}$ satisfies $d^{(n)} = 2^{-n}d^{(0)}$ which converges to 0 as $n \to \infty$. Note that f is differentiable in the in the complex sense at z_0, so there exists a function $\phi : A \to \mathbb{C}$, continuous at z_0 with

$$f(z) = f(z_0) + (z - z_0)\phi(z), \quad \text{and } \phi(z_0) = f'(z_0).$$

Note that for $\epsilon > 0$, there exists $\delta > 0$, such that

$$|\phi(z) - f'(z_0)| < \epsilon, \quad \forall z \in D(z_0, \delta) \cap A.$$

Now, for $n \geq N$, $\triangle^{(n)} \subseteq D(z_0, \delta)$ and thus

$$\begin{aligned}
\int_{\gamma_{\partial\triangle^{(n)}}} f(z)dz &= \int_{\gamma_{\partial\triangle^{(n)}}} (f(z_0) + f'(z_0)(z - z_0) + (\phi(z) - f'(z_0))(z - z_0))dz \\
&= \int_{\gamma_{\partial\triangle^{(n)}}} f(z_0)dz + \int_{\gamma_{\partial\triangle^{(n)}}} f'(z_0)(z - z_0)dz + \int_{\gamma_{\partial\triangle^{(n)}}} (\phi(z) - f'(z_0))(z - z_0)dz \\
&= f(z_0)\int_{\gamma_{\partial\triangle^{(n)}}} dz + f'(z_0)\int_{\gamma_{\partial\triangle^{(n)}}} (z - z_0)dz + \int_{\gamma_{\partial\triangle^{(n)}}} (\phi(z) - f'(z_0))(z - z_0)dz \\
&= \int_{\gamma_{\partial\triangle^{(n)}}} (\phi(z) - f'(z_0))(z - z_0)dz \quad \text{since the first two integrands both have primitives on } A.
\end{aligned}$$

Then, we have for $n \geq N$ that

$$\left|\int_{\gamma_{\partial\triangle^{(n)}}} f(z)dz\right| \leq \int_{\gamma_{\partial\triangle^{(n)}}} |\phi(z) - f'(z_0)|\,|z - z_0|\,dz \leq \epsilon d^{(n)} \int_{\gamma_{\partial\triangle^{(n)}}} ds = \epsilon d^{(n)} l(\gamma_{\partial\triangle^{(n)}}).$$

As a result,

$$\left|\int_{\gamma_{\partial\triangle}} f(z)dz\right| \leq 4^n \epsilon d^{(n)} l(\gamma_{\partial\triangle^{(n)}}) = \epsilon \cdot (2^n d^{(n)}) \cdot (2^n l(\gamma_{\partial\triangle^{(n)}})) = \epsilon d^{(0)} l(\gamma_{\partial\triangle}),$$

and by arbitrariness of ϵ, we can conclude the desired result. $\square$

Definition 4.21. An open set $A \subseteq \mathbb{C}$ is called a **star-shaped domain** if there is a point $z_\star \in A$ such that for any $z \in A$, also the connecting path between $z_\star$ and z is contained in A, that is,

$$\{\lambda z + (1 - \lambda)z_\star;\ \lambda \in [0, 1]\} \subseteq A.$$

The point $z_\star \in A$ is called a **star center** of A.

Note that star-shaped domains are clearly connected, so the name "domain" is justified.

Example 4.22. (1) $\mathbb{C}$ is a star-shaped domain, and every point $z \in \mathbb{C}$ is a star center.

(2) Nonempty, convex, open subsets A of $\mathbb{C}$ are star-shaped domains, and every point $z \in A$ is a star center.

(3) $\mathbb{C}_- = \mathbb{C}\setminus\{x \in \mathbb{R};\ x \leq 0\}$ is star-shaped, and every point on the positive real line is a star center.

(4) $\mathbb{C}\setminus\{0\}$, annuli $D(z_0, R)\setminus\overline{D(z_0, r)}$ where $0 < r < R$, and deleted disks $\dot{D}(z_0, R)$ where $R > 0$ are not star-shaped domains.

Theorem 4.23 (Cauchy's integral theorem)**.** Let $A \subseteq \mathbb{C}$ be a star-shaped domain and $f : A \to \mathbb{C}$ holomorphic. Then the following statements hold:

(1) There is a primitive of f on A, *i.e.*, there exists $F : A \to \mathbb{C}$ holomorphic with

$$F'(z) = f(z), \quad \text{for } z \in A.$$

(2) Integrals over f are path independent, *i.e.*, for a piecewise continuously differentiable curve $\gamma : I = [a, b] \to A$ (where $a < b$) with $\gamma(I) \subseteq A$, the integral $\int_\gamma f(z)dz$ depends only on $\gamma(b)$ and $\gamma(a)$.

(3) Integrals over f along closed curves are zero, *i.e.*, if $\gamma : I = [a, b] \to \mathbb{C}$ (where $a < b$) is a piecewise continuously differentiable curve with $\gamma(I) \subseteq A$ and $\gamma(a) = \gamma(b)$, then $\int_\gamma f(z)dz = 0$.

Proof. We have already proved that the three statements are equivalent at the beginning of this lecture, so it suffices to establish (1). Let $z_\star \in A$ be a star center and denote for $z \in A$ the line segment connecting $z_\star$ and z by $\gamma_{[z_\star,z]} : I = [0,1] \to \mathbb{C}$. Clearly, $\gamma_{[z_\star,z]}(I) \subseteq A$, since A is star-shaped. Now we define

$$F(z) = \int_{\gamma_{[z_\star,z]}} f(w)dw.$$

For $\delta > 0$ small enough, since A is open, there exists some $\delta > 0$ such that $D(z,\delta) \subseteq A$. Then for any $w \in D(z,\delta)$, every point on the line segment $[z,w]$ connecting z and w is in A. Since A is star-shaped, the line segments $[z,z_\star]$ and $[w,z_\star]$ are also contained in A, so that $\triangle(z_\star,w,z) \subseteq A$. Now by Goursat's Lemma, we can obtain that

$$0 = \int_{\gamma_{\partial\triangle(z_\star,w,z)}} f(u)du = \int_{\gamma_{[z_\star,w]}} f(u)du + \int_{\gamma_{[w,z]}} f(u)du + \int_{\gamma_{[z,z_\star]}} f(u)du = F(w) + \int_{\gamma_{[w,z]}} f(u)du - F(z).$$

Now fix $\epsilon > 0$, then since f is continuous at z, we can find some $\delta' > 0$, such that for all $u \in D(z,\delta') \cap A$, we have $|f(u) - f(z)| < \epsilon$. Take $\delta'' = \min\{\delta,\delta'\}$, then for any $w \in \dot{D}(z,\delta'') \subseteq A$, we have that

$$\left|\frac{F(w)-F(z)}{w-z} - f(z)\right| = \left|\frac{1}{w-z}\int_{\gamma_{[z,w]}} f(u)du - f(z)\right| \leq \frac{1}{|w-z|}\int_{\gamma_{[z,w]}} |f(u)-f(z)|\,du < \frac{\epsilon l(\gamma_{[z,w]})}{|w-z|} = \epsilon,$$

thus we can conclude that F is differentiable in the complex sense at $z \in A$ and that $F'(z) = f(z)$ by arbitrariness of ϵ, and the proof is done. □

The integral theorem can be used to simplify significantly the calculation of certain path integrals by deformation, as we can see in the following example.

Example 4.24. Let $\xi \in \mathbb{R}$. We consider the function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \exp(-\pi x^2)$ and we aim at computing its Fourier transform[4]

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \exp(-\pi x^2)\exp(-2\pi i x\xi)dx.$$

We recall from real analysis that

$$\int_{-\infty}^{\infty} \exp(-\pi x^2)dx = 1,$$

and by noticing that

$$\exp(-\pi x^2)\exp(-2\pi i x\xi) = \exp(-\pi\xi^2)\exp(-\pi(x+i\xi)^2),$$

we claim and prove by Cauchy's integral theorem for any $\xi \in \mathbb{R}$ that

$$\int_{-\infty}^{\infty} \exp(-\pi(x+i\xi)^2)dx = 1.$$

Consider the following rectangle

$$Q_R = \{z \in \mathbb{C};\ |\mathrm{Re}(z)| \leq R,\ 0 \leq \mathrm{Im}(z) \leq \xi\},$$

where $\xi, R > 0$. Its boundary ∂Q_R, when traversed in a counterclockwise direction, can be parametrized as $\gamma_{\partial Q_R} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, where

$$\begin{aligned}
\gamma_1 &: [0,1] \to \mathbb{C}, \quad t \mapsto 2Rt - R,\\
\gamma_2 &: [1,2] \to \mathbb{C}, \quad t \mapsto R + i(t-1)\xi,\\
\gamma_3 &: [2,3] \to \mathbb{C}, \quad t \mapsto R - 2R(t-2) + i\xi,\\
\gamma_4 &: [3,4] \to \mathbb{C}, \quad t \mapsto -R + i(4-t)\xi,
\end{aligned}$$

as is shown in Figure 3. The function $f : z \mapsto \exp(-\pi z^2)$ is holomorphic on the star-shaped domain $\mathbb{C}$ which contains

[4]There are other possible definitions of the Fourier transform.

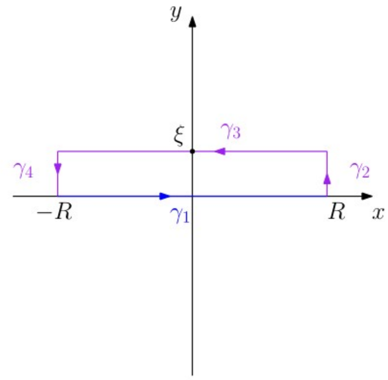

Figure 3: The partition $\gamma_{\partial Q_R} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$

$\gamma_{\partial Q_R}$, so by Cauchy's Integral Theorem, we have that

$$\begin{aligned} 0 = \int_{\gamma_{\partial Q_R}} f(z)dz &= \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz + \int_{\gamma_4} f(z)dz \\ &=: \int_{-R}^{R} \exp(-\pi x^2)dx + I_2 - \int_{-R}^{R} \exp(-\pi(x+i\xi)^2)dx + I_4. \end{aligned}$$

Now that

$$|I_2| = \left|\int_{\gamma_2} f(z)dz\right| = \left|\int_1^2 \exp(-\pi(R+i(t-1)\xi)^2)\cdot i\xi dt\right| \leq C\exp(-\pi R^2) \to 0 \quad \text{as } R\to\infty,$$

since ξ is fixed and is negligible when bringing $R\to\infty$. Similarly we have that

$$|I_4| = \left|\int_{\gamma_4} f(z)dz\right| = \left|\int_3^4 \exp(-\pi(-R+i(4-t)\xi)^2)\cdot(-i\xi)\right| \leq C\exp(-\pi R^2) \to 0 \quad \text{as } R\to\infty.$$

Hence, bringing $R\to\infty$, we obtain that

$$\int_{-\infty}^{\infty} \exp(-\pi(x+i\xi)^2)dx = \int_{-\infty}^{\infty} \exp(-\pi x^2)dx + \lim_{R\to\infty} I_2 + \lim_{R\to\infty} I_4 = 1.$$

The case is similar for $\xi < 0$. Therefore, we obtain that

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \exp(-\pi x^2)\exp(-2\pi i x\xi)dx = \exp(-\pi\xi^2)\int_{-\infty}^{\infty} \exp(-\pi(x+i\xi)^2)dx = \exp(-\pi\xi^2).$$

10/24 Lecture

(Continued)

4.3 Homotopies and Simply Connected Domains

Definition 4.25. Let $A \subseteq \mathbb{C}$ be open and $\gamma_0, \gamma_1 : I = [0,1] \to \mathbb{C}$ two continuous curves with $\gamma_0(I), \gamma_1(I) \subseteq A$ and

$$\gamma_0(0) = \gamma_1(0), \quad \gamma_0(1) = \gamma_1(1).$$

We say that γ_0 and γ_1 are **homotopic with fixed endpoints** in A if there is a continuous function $H : [0,1]\times[0,1] \to \mathbb{C}$ with $H([0,1]^2) \subseteq A$ fulfilling

(1) $H(0,t) = \gamma_0(t)$ and $H_1(t) = \gamma_1(t)$ for every $t \in [0,1]$;

(2) For every $s \in [0,1]$, it holds that

$$H(s,0) = \gamma_0(0) = \gamma_1(0), \quad H(s,1) = \gamma_0(1) = \gamma_1(1).$$

The function H is called a **homotopy** between γ_0 and γ_1.

The intuitive meaning of the definition is that as s ranges from 0 to 1, the curve $H(0,\cdot) = \gamma_0$ is deformed in a continuous manner into $H(1,\cdot) = \gamma_1$.

Remark 4.26. (1) Homotopy with fixed endpoints is an equivalence relation.

(2) The fact that γ_0, γ_1 are defined on $I = [0,1]$ is no restriction, since every continuous curve $\gamma : \tilde{I} = [a,b] \to \mathbb{C}$ with $a < b$ can be parametrized to be defined on $I = [0,1]$.

(3) The construction $H(s,t) = s\gamma_1(t) + (1-s)\gamma_0(t)$ works in general if the domain A in which the curves γ_0 and γ_1 are defined is convex.

(4) In the definition of homotopy we purposefully do not rule out the case of $\gamma_0(0) = \gamma_1(0) = \gamma_0(1) = \gamma_1(1)$. This special case gives homotopy of closed curves fixed at a point[5].

Definition 4.27. (1) Let $A \subseteq \mathbb{C}$ be open and $\gamma : I = [0,1] \to \mathbb{C}$ with $\gamma(I) \subseteq A$ a closed continuous curve with $z_0 = \gamma(0) = \gamma(1)$. We say that γ is **null-homotopic** (in A) if it is homotopic (in A) to the constant curve $\gamma_{\{z_0\}} : [0,1] \to \mathbb{C}$ with $\gamma_{\{z_0\}}(t) = z_0$ for all $t \in [0,1]$.

(2) A given domain A is **simply connected** if every closed curve in A is null-homotopic.

Intuitively, a simply connected domain is a connected open set with no holes. Otherwise, a closed curve surrounding that hole will not be null-homotopic.

Remark 4.28. (1) A domain A is simply connected if and only if every two continuous curves $\gamma_0, \gamma_1 : I = [0,1] \to \mathbb{C}$ in A with $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$ are homotopic with fixed endpoints in A.

(2) Convex domains are simply connected. More generally, star-shaped domains are simply connected.

We will prove a version of Cauchy's integral theorem on simply connected domains. We need two lemmas as preparation.

Lemma 4.29. Let $A \subseteq \mathbb{C}$ be open and γ_0 and γ_1 be homotopic continuously differentiable curves with fixed endpoints in A. Then there exists a homotopy $H : [0,1] \times [0,1] \to \mathbb{C}$ between γ_0 and γ_1 in A (*i.e.*, with $H([0,1]^2) \subseteq A$) with $H \in \mathbb{C}^1([0,1] \times [0,1]; \mathbb{C})$. We say that H is a C^1-homotopy between γ_0 and γ_1.

The proof of the above lemma is technical and relies on mollification of continuous functions, which we will not explicitly show here.

Lemma 4.30. Let $K \subseteq A \subseteq \mathbb{C}$ with A open and K compact and non-empty. Then there is a number $r > 0$ such that

$$d(K, \partial A) = \inf\{|z - w|;\ z \in K, w \in \partial A\} > r.$$

Proof. For every $z \in K$, take $r_z > 0$ with $D(z, 2r_z) \subseteq A$. Clearly $\{D(z, r_z)\}_{z \in K}$ is an open cover of K. By compactness, there is a finite subcover

$$K \subseteq \bigcup_{v=1}^{N} D(z_v, r_{z_v}).$$

Set $\rho = \min_{v \in \{1,\cdots,N\}} r_{z_v}$. For every $z \in K$, one has some $v \in \{1,\cdots,N\}$ with $|z - z_v| < r_{z_v}$. Then for $w \in D(z,\rho)$, we have that

$$|w - z_v| \leq |w - z| + |z - z_v| < \rho + r_{z_v} \leq 2r_{z_v}.$$

Hence, $w \in D(z_v, 2r_{z_v}) \subseteq A$, and so $D(z,\rho) \subseteq A$. Setting $r = \rho/2$, then for $z \in K$ and $w \in \partial A$ we would have that $|z - w| \geq \rho > r$, which completes the proof. $\square$

Now we are ready to state Cauchy's Integral Theorem for simply connected domains.

[5] More generally, one could also define (free) homotopy of two closed curves γ_0 and γ_1 in A by replacing the second condition in the definition of homotopies by the condition $H(s,0) = H(s,1)$ for every $s \in [0,1]$. In this case, the start- and endpoints of γ_0 and γ_1 do not need to coincide.

Theorem 4.31 (Cauchy's integral theorem for simply connected domains)**.** Let $A \subseteq \mathbb{C}$ be a simply connected domain and $f : A \to \mathbb{C}$ be holomorphic. Then the following statements hold:

(1) There is a primitive of f on A, *i.e.*, there exists $F : A \to \mathbb{C}$ holomorphic with

$$F'(z) = f(z), \quad \text{for } z \in A.$$

(2) Integrals over f are path independent, *i.e.*, for a piecewise continuously differentiable curve $\gamma : I = [a, b] \to A$ (where $a < b$) with $\gamma(I) \subseteq A$, the integral $\int_\gamma f(z)dz$ depends only on $\gamma(b)$ and $\gamma(a)$.

(3) Integrals over f along closed curves are zero, *i.e.*, if $\gamma : I = [a, b] \to \mathbb{C}$ (where $a < b$) is a piecewise continuously differentiable curve with $\gamma(I) \subseteq A$ and $\gamma(a) = \gamma(b)$, then $\int_\gamma f(z)dz = 0$.

Proof. We have already proved that the three statements are equivalent, so we only need to establish (ii). First assume that γ_0 and γ_1 are homotopic continuously differentiable curves and let $H \in C^1([0, 1]^2; \mathbb{C})$ be a C^1-homotopy in A between γ_0 and γ_1, which exists by the first preparing lemma. Note that $K = H([0, 1]^2) \subseteq A$ is a compact set. Hence there is $r > 0$ with $d(K, \partial A) > r$ by the second preparing lemma, and there are finitely many open disks $D(z_j, r)$, $1 \le i \le n$ with $z_j \in K$ and $\overline{D(z_j, r)} \subseteq A$ over K. Indeed, $K \subseteq \bigcup_{z\in K} D(z, r)$, so $\{D(z, r)\}_{z\in K}$ is an open cover of the compact set K that must admit a finite open subcover, and since $d(K, \partial A) > r$, we must have $\overline{D(z, r)} \subseteq A$ for every $z \in K$. Now we define $\gamma_s(t) := H(s, t)$ for $(s, t) \in [0, 1]^2$, so the curves $\gamma_s : [0, 1] \to \mathbb{C}$, $t \mapsto \gamma_s(t)$ are continuously differentiable curves. We then define

$$F : [0, 1] \to \mathbb{C}, \quad s \mapsto \int_{\gamma_s} f(z)dz.$$

We will now show that F is constant on $[0, 1]$. Define $\tilde{I} = \{s \in [0, 1];\ F(s) = F(0)\}$. We show that $\tilde{I} \neq \varnothing$, and is both closed and open (in the induced topology of $[0, 1]$).

- We show that $\tilde{I} \neq \varnothing$. Indeed, $0 \in \tilde{I}$.
- We show that $\tilde{I}$ is closed. Indeed, the map

  $$s \mapsto F(s) = \int_0^1 f(\gamma_s(t))\gamma_s'(t)dt$$

  is continuous, since the map $(s, t) \mapsto f(\gamma_s(t))\gamma_s'(t)$ is uniformly continuous. Since $\tilde{I}$ is the inverse image of the closed set $\{F(0)\}$ under this map, it must also be closed.
- We show that $\tilde{I}$ is open. Equivalently, let $s_0 \in \tilde{I}$ and we show that there exists $\delta > 0$ such that $(s_0 - \delta, s_0 + \delta) \cap [0, 1] \subseteq \tilde{I}$. Indeed, there exists $1 \le J \le n$ such that

  $$\gamma_{s_0}([0, 1]) \subseteq \bigcup_{j=1}^{J} D(z_j, r),$$

  and for a partition $0 = t_0 < t_1 < \cdots < t_J = 1$, one has

  $$\gamma_{s_0}([t_{j-1}, t_j]) \subseteq D(z_j, r), \quad \text{for } 1 \le j \le J.$$

  Note that

  $$\gamma_{s_0}(t_j) \subseteq D(z_j, r) \cap D(z_{j+1}, r), \quad \text{for } 1 \le j \le J - 1.$$

  Since $D(z_j, r)$ is open for every $1 \le j \le J$, there exists $\delta > 0$ such that the previous two conditions hold[6] for s_0 replaced by an arbitraty $s \in [0, 1]$ with $|s - s_0| \le \delta$. Now consider for $1 \le j \le J$ the closed and piecewise continuously differentiable curves

  $$\Gamma_j = \gamma_s|_{[t_{j-1},t_j]} + \xi_j - \gamma_{s_0}|_{[t_{j-1},t_j]} + \xi_{j-1},$$

  where ξ_j denotes the straight line segment between $\gamma_s(t_j)$ and $\gamma_{s_0}(t_j)$. By convention, ξ_0 is the constant curve in the point $\gamma_{s_0}(0) = \gamma_s(0)$ and ξ_J is the constant curve in the point $\gamma_{s_0}(1) = \gamma_s(1)$. We visualize this construction

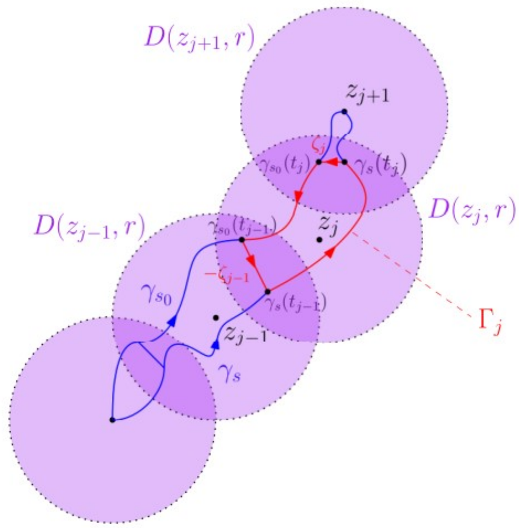

Figure 4: The elements of the construction of Γ_j

in Figure 4. By construction, the image of Γ_j is contained in $D(z_j, r)$, which is a star-shaped domain since it is convex. We therefore apply Cauchy's integral theorem for star-shaped domains, which implies that

$$\int_{\Gamma_j} f(z)dz = 0, \quad \text{for } i \leq j \leq J.$$

Therefore, we can see that

$$F(s) - F(s_0) = \int_{\gamma_s} f(z)dz - \int_{\gamma_{s_0}} f(z)dz = \sum_{j=1}^{J} \int_{\Gamma_j} f(z)dz = 0.$$

This shows that $\tilde{I}$ is open.

Now since $[0,1]$ is connected, we must have $\tilde{I} = [0,1]$ and we conclude the proof in the case of continuously differentiable curves γ_0 and γ_1. Now assume that γ_0 is only piecewise continuously differentiable, with $0 = t_0 < t_1 < \cdots < t_K = 1$ and $\gamma_0|_{[t_{k-1},t_k]} \in C^1([t_{k-1}, t_k]; \mathbb{C})$. One can replace γ_0 in the disks $D(\gamma_0(t_k), R_k) \subseteq A$, $1 \leq k \leq K-1$ by a continuously differentiable curve ξ_k staying within $D(\gamma_0(t_k), R_k)$ with

$$\xi_k(\pm 1) = \gamma_0(t_k \pm \epsilon), \quad \xi_k'(\pm 1) = \gamma_0'(t_k \pm \epsilon) \quad \text{for } 1 \leq k \leq K-1,$$

as is shown in Figure 5.

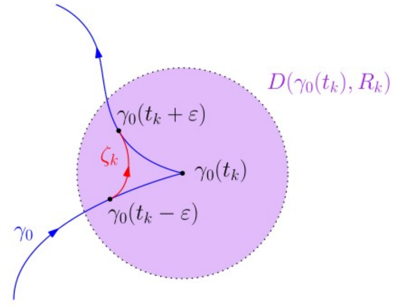

Figure 5: Replacing the curve $\gamma_0|_{[t_k-\epsilon,t_k+\epsilon]}$ by ξ_k

[6]Here, one has to uuse that H is uniformly continuous (as a continuous function on the compact set $[0,1] \times [0,1]$).

Applying again Cauchy's integral theorem for star-shaped domains, we have that

$$\int_{\xi_k} f(z)dz = \int_{\gamma_0|_{[t_k-\epsilon, t_k+\epsilon]}} f(z)dz \quad \text{for } 1 \leq k \leq K-1.$$

The curve $\tilde{\gamma_0}$ obtained by replacing all $\gamma_0|_{[t_k-\epsilon,t_k+\epsilon]}$ by ξ_k is then a continuously differentiable curve and does not change the value of the integral. This is then back to the previous case and we are done. ☐

The theorem above shows that if $A \subseteq \mathbb{C}$ is a simply connected domain, then every holomorphic function $f : A \to \mathbb{C}$ has a primitive on A. In fact, the converse is also true and can be shown using conformal mappings and the **Riemann mapping theorem**. In other words, simply connected domains are **exactly** the domains, in which every holomorphic function has a primitive.

4.4 Cauchy's Integral Formula

We will now state and prove Cauchy's integral formula, which will follow using Cauchy's integral theorem for star-shaped domains. This powerful formula will be instrumental in proving that every holomorphic function is differentiable in the complex sense infinitely often.

Theorem 4.32 (Cauchy's integral formula)**.** Let $A \subseteq \mathbb{C}$ be open and $f : A \to \mathbb{C}$ holomorphic. Let $r > 0$ and $z_0 \in A$ such that

$$\overline{D(z_0, r)} = \{z \in \mathbb{C};\ |z - z_0| \leq r\} \subseteq A.$$

Then for all $z \in D(z_0, r)$, Cauchy's integral formula holds:

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(w)}{w - z} dw,$$

where $\partial D(z_0, r)$ is any parametrization of the curve running along the boundary of $D(z_0, r)$ in a counterclockwise direction, for instance $\gamma_{\partial D(z_0,r)}(t) = z_0 + r \exp(2\pi i t)$, $t \in [0, 1]$.

Proof. Let $z \in D(z_0, r)$ be fixed and z_1 be one of the intersections of the line connecting z to z_0 with $\partial D(z_0, r)$ (if $z = z_0$, then any point on $\partial D(z_0, r)$ will work). Furthermore, let $\epsilon > 0$ be chosen such that $\overline{D(z, \epsilon)} \subseteq D(z_0, r)$. We define the closed piecewise continuously differentiable curve γ_1 starting at z_1, moving in a counterclockwise fashion along the half circle, the first part of the diameter, the smaller the smaller half-circle in a clockwise fashion and finally the second part of the diameter. Finally, we let ξ_ϵ be a parametrization of $\partial D(z, \epsilon)$ in a counterclockwise fashion. The scheme is as shown in Figure 6.

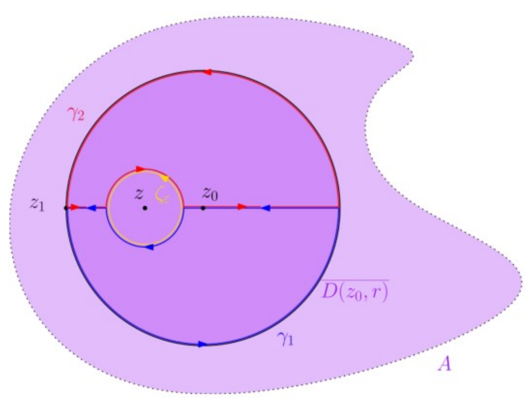

Figure 6: The construction of γ_1, γ_2, and ξ_ϵ

The images of both curves γ_1 and γ_2 are in star-shaped domains, obtained by deleting from $D(z_0, r + \delta) \subseteq A$ for some $\delta > 0$ a half-line at z orthogonal to the line connecting z_1 and z_0, on which the function

$$w \mapsto \frac{f(w)}{w - z} \in \mathbb{C}$$

is holomorphic. Therefore, Cauchy's integral theorem for star-shaped domains applies and we obtain

$$0 = \int_{\gamma_1} \frac{f(w)}{w-z} dw + \int_{\gamma_2} \frac{f(w)}{w-z} dw = \int_{\partial D(z_0,r)} \frac{f(w)}{w-z} dw + \int_{-\gamma_\epsilon} \frac{f(w)}{w-z} dw.$$

We therefore obtain

$$\int_{\partial D(z_0,r)} \frac{f(w)}{w-z} dw = f(z) \int_{\gamma_\epsilon} \frac{1}{w-z} dw + \int_{\gamma_\epsilon} \frac{f(w)-f(z)}{w-z} dw.$$

The first integral can be easily calculated by using the parametrization $\gamma_\epsilon(t) = z + \epsilon \exp(2\pi i t)$, $t \in [0,1]$, so that

$$\int_{\gamma_\epsilon} \frac{1}{w-z} dw = \int_0^1 \frac{2\pi i \epsilon \exp(2\pi i t)}{\epsilon \exp(2\pi i t)} dt = 2\pi i.$$

For the second integral, since f is holomorphic at $w = z$, it is also continuous at that point, and thus for arbitrary $\epsilon' > 0$ we can find $\delta > 0$ such that

$$|f(w) - f(z)| < \epsilon', \quad \text{for } w \in D(z,\delta) \cap A.$$

Therefore, we can obtain that

$$\left| \int_{\gamma_\epsilon} \frac{f(w)-f(z)}{w-z} dw \right| \leq l(\gamma_\epsilon) \cdot \frac{\epsilon'}{\epsilon} = 2\pi\epsilon',$$

and by arbitrariness of $\epsilon' > 0$, this vanishes. Hence, we can conclude that

$$\int_{\partial D(z_0,r)} \frac{f(w)}{w-z} dw = 2\pi i f(z) \implies f(z) = \frac{1}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(w)}{w-z} dw.$$

□

The Cauchy integral formula allows us to calculate the values of a holomorphic function in the interior of a disk from their values on the boundary. In fact, by interchanging differentiation and integration, we can show similar formulas for the derivatives of a holomorphic function. The formal justification is the following **Leibniz rule**.

Lemma 4.33 (Leibniz rule)**.** Let $a < b \in \mathbb{R}$ and $A \subseteq \mathbb{C}$ open. Consider a continuous function $f : [a,b] \times A \to \mathbb{C}$, such that $f(t,\cdot)$ is holomorphic for every fixed $t \in [a,b]$, and $(t,z) \mapsto \frac{\partial f}{\partial z}(t,z)$ is continuous on $[a,b] \times A$. Then the function $g : A \mapsto \mathbb{C}$, defined by

$$g(z) = \int_a^b f(t,z) dt,$$

is holomorphic and one has that

$$g'(z) = \int_a^b \frac{\partial f}{\partial z}(t,z) dt.$$

Theorem 4.34. Let $A \subseteq \mathbb{C}$ be open and $f : A \to \mathbb{C}$ holomorphic. Then the following statements hold:

(1) f has arbitrarily many complex derivatives in A.

(2) Let $r > 0$ and $z_0 \in A$ such that $\overline{D(z_0,r)} \subseteq A$. Then for every $n \in \mathbb{N}$ and $z \in D(z_0,r)$, we have the generalized version of Cauchy's integral formula:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D(z_0,r)} \frac{f(w)}{(w-z)^{n+1}} dw,$$

where $\partial D(z_0,r)$ is a parametrization of the boundary of $D(z_0,r)$ as in Cauchy's integral formula.

Proof. Assume the conditions of (2). By Cauchy's integral formula, we have that

$$f(z) = \frac{1}{2\pi i}\int_{\partial D(z_0,r)} \frac{f(w)}{w-z}dw \quad \text{for } z \in D(z_0,r).$$

Now the function $z \mapsto \frac{f(w)}{w-z}$ is continuously differentiable with

$$\frac{\partial}{\partial z}\left(\frac{f(w)}{w-z}\right) = \frac{f(w)}{(w-z)^2}.$$

So by Leibniz rule, we can differentiate with respect to z (after inserting a parametrization of $\partial D(z_0,r)$) to obtain

$$f'(z) = \frac{1}{2\pi i}\int_{\partial D(z_0,r)} \frac{\partial}{\partial z}\left(\frac{f(w)}{w-z}\right)dw = \frac{1}{2\pi i}\int_{\partial D(z_0,r)} \frac{f(w)}{(w-z)^2}dw \quad \text{for } z \in D(z_0,r).$$

The claim now follows by induction on $n \in \mathbb{N}$. Indeed, the case $n = 1$ is just shown, and assume for $n \in \mathbb{N}$, we will have for $(n+1)$ that

$$f^{(n+1)}(z) = \frac{d}{dz}f^{(n)}(z) = \frac{d}{dz}\frac{n!}{2\pi i}\int_{\partial D(z_0,r)} \frac{f(w)}{(w-z)^{n+1}}dw = \frac{(n+1)n!}{2\pi i}\int_{\partial D(z_0,r)} \frac{f(w)}{(w-z)^{n+2}}dw,$$

which completes the inductive step. The claim (1) now follows since for $z_0 \in A$, there is an $r > 0$ such that $\overline{D(z_0,r)} \subseteq A$, and $f|_{D(z_0,r)}$ has arbitrarily many derivatives. □

4.5 Consequences of Cauchy's Integral Formulas

In this section, we state and prove some applications of the Cauchy integral formulas. In particular, we will prove Liouville's theorem and infer from it the Fundamental Theorem of Algebra. The first ingredient of Liouville's theorem are the following Cauchy inequalities.

Proposition 4.35. Let $z_0 \in \mathbb{C}$, $r > 0$, and $f : D(z_0,r) \to \mathbb{C}$ holomorphic. If there is a number $M > 0$ with $|f(z)| \leq M$ for all $z \in D(z_0,r)$, then for every $n \in \mathbb{N}$, we have that

$$|f^{(n)}(z_0)| \leq M\frac{n!}{r^n}.$$

Proof. Take the generalized version of Cauchy's integral formula with $z = z_0$ and $\overline{D(z_0,\rho)} \subseteq D(z_0,r)$ for any $0 < \rho < r$. Note that for $w \in \partial D(z_0,\rho)$, we have that $|z_0 - w| = \rho$, and thus

$$\left|\frac{f(w)}{(w-z_0)^{n+1}}\right| \leq \frac{M}{\rho^{n+1}} \quad \text{for } w \in \partial D(z_0,\rho).$$

Therefore, we have that

$$|f^{(n)}(z_0)| = \left|\frac{n!}{2\pi i}\int_{\partial D(z_0,\rho)} \frac{f(w)}{(w-z_0)^{n+1}}dw\right| \leq \frac{n!}{2\pi}\frac{M}{\rho^{n+1}}l(\gamma_{\partial D(z_0,r)}) = M\frac{n!}{\rho^n}.$$

By arbitrariness of $\rho \in (0,r)$, bringing $\rho \to r^-$ we conclude our proof. □

Theorem 4.36 (Liouville's theorem)**.** A bounded entire function $f : \mathbb{C} \to \mathbb{C}$ is constant.

Proof. Let f be bounded, so there exists $C > 0$ such that $|f| \leq C$. Then for any $z_0 \in \mathbb{C}$ and $r > 0$, by the previous proposition we have that

$$|f'(z_0)| \leq \frac{C}{r}.$$

Bringing $r \to \infty$, we thus observe that $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$, and thus f must be constant. □

Theorem 4.37 (Fundamental Theorem of Algebra)**.** Let $a_0, \cdots, a_n \in \mathbb{C}$, with $n \in \mathbb{N}$ and $a_n \neq 0$. Then the polynomial

$$P : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \sum_{v=0}^{n} a_v z^v$$

has a zero, *i.e.*, there exists a point $z_0 \in \mathbb{C}$ with $P(z_0) = 0$.

Proof. Assume that $P(z) \neq 0$ for all $z \in \mathbb{C}$. THen the function $z \mapsto \frac{1}{P(z)}$ would be an entire function. Now consider for $z \neq 0$ that

$$P(z) = z^n \left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right).$$

Clearly, $\lim_{z \to \infty} \left(\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right) = 0$, so there exists $K > 0$, such that

$$\left| \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right| < \frac{|a_n|}{2} \quad \text{for } |z| > K.$$

By the inverse triangle inequality, we see that

$$|P(z)| = |z|^n \left| a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right| \geq |z|^n \left(|a_n| - \left| \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right| \right) > |z|^n \frac{|a_n|}{2}.$$

Since $a_n \neq 0$ and $n \geq 1$, $P(z) \to \infty$ as $|z| \to \infty$, and so for $M > 0$, there exists $R > 0$, such that

$$|P(z)| \geq M \implies \left| \frac{1}{P(z)} \right| \leq \frac{1}{M}, \quad \text{for } |z| > R.$$

By assumption, $z \mapsto \frac{1}{P(z)}$ is entire, so it is continuous, and thus bounded on the compact set $\overline{D(0, R)}$. In total, $z \mapsto \frac{1}{P(z)}$ would be a bounded entire function, so by Liouville's theorem it must be constant and so is P, which leads to a contradiction. Therefore, P must have a zero. $\square$

As another application of Cauchy's integral formula, we show **Morera's theorem**, which is in a way a converse to Goursat's Lemma (Lemma 4.19).

Theorem 4.38 (Morera's theorem)**.** Let $A \subseteq \mathbb{C}$ open and $f : A \to \mathbb{C}$ a continuous function. Assume that for every $\triangle(z_1, z_2, z_3) \subseteq A$, one has that

$$\int_{\gamma_{\partial \triangle(z_1, z_2, z_3)}} f(z) dz = 0.$$

Then f is holomorphic on A.

Proof. For every $z_0 \in A$, there exists $\epsilon > 0$, such that $D(z_0, \epsilon) \subseteq A$. It suffices to show that $f|_{D(z_0, \epsilon)}$ is holomorphic. We define on the star-shaped domain $D(z_0, \epsilon)$ the function

$$F(z) = \int_{\gamma_{[z_0, z]}} f(w) dw.$$

As in the proof of Cauchy's theorem for star-shaped domains, F is a primitive of $f|_{D(z_0, \epsilon)}$, meaning that it is holomorphic and fulfills $F'(z) = f(z)$ for every $z \in D(z_0, \epsilon)$. By the generalized form of Cauchy's integral formula, F has infinitely many complex derivatives, and in particular, f is holomorphic. $\square$

11/7 Lecture

5 Maximum Modulus Theorem and Harmonic Functions

5.1 Maximum Modulus Theorem

The following proposition is the **mean value property** of holomorphic functions.

Proposition 5.1. Let $A \subseteq \mathbb{C}$ be open and $f : A \to \mathbb{C}$ holomorphic. Let $r > 0$ and $z_0 \in A$ such that $\overline{D(z_0, r)} \subseteq A$. Then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r\exp(it))dt.$$

Proof. This follows immediately from Cauchy's integral formula. Indeed,

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma_{\partial D(z_0,r)}} \frac{f(w)}{w - z_0} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + r\exp(it))}{r\exp(it)} ir\exp(it)dt = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r\exp(it))dt.$$

□

Therefore, we can see that the value of a holomorphic function at a point agrees with its "circular average" around that point. We use this fact to show the **local maximum modulus principle**.

Theorem 5.2 (Local maximum modulus principle)**.** Let $A \subseteq \mathbb{C}$ be open.and $f : A \to \mathbb{C}$ holomorphic. Assume that for $r > 0$ and $z_0 \in A$ with $D(z_0, r) \subseteq A$, we have that

$$|f(z_0)| \geq |f(z)|, \qquad \text{for } z \in D(z_0, r).$$

Then f is constant on $D(z_0, r)$.

Proof. Let $0 < \delta < r$, then by the mean value property, we obtain that

$$|f(z_0)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \delta\exp(it))dt \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \delta\exp(it))|dt.$$

Since by assumption, for all $t \in [0, 2\pi]$ we have $|f(z_0 + \delta\exp(it))| \leq |f(z_0)|$, it follows that

$$\frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \delta\exp(it))|dt \leq \frac{|f(z_0)|}{2\pi} \int_0^{2\pi} dt = |f(z_0)|.$$

Therefore, all the inequalities should take their equal condition, which means that $|f(z_0 + \delta\exp(it))| = |f(z_0)|$ for all $t \in [0, 2\pi]$, so we conclude that $|f(z_0)| = |f(z)|$ for $z \in \partial D(z_0, \delta)$. But since $D(z_0, r) = \{z_0\} \cup \bigcup_{\delta \in (0,r)} \partial D(z_0, \delta)$, we obtain that $|f|$ has the constant value $|f(z_0)|$ on all of $D(z_0, r)$. Since $f|_{D(z_0,r)}$ is a holomorphic function with constant modulus on the connected set $D(z_0, r)$, it must be constant itself. □

Therefore, a holomorphic function cannot attain a local maximum in its modulus, unless it is constant in a neighborhood of the point where the maximum is attained.

Remark 5.3.

(1) Note a holomorphic function can attain a minimum in its modulus at zeros, without being zero is a neighborhood.

(2) However, assume that a holomorphic function $f : A \to \mathbb{C}$ does not have zeros in $D(z_0, r) \subseteq A$ and $|f(z_0)| \leq |f(z)|$ for all $z \in D(z_0, r)$ (*i.e.*, z_0 is a local minimum in modulus of f). We can apply the maximum modulus principle to the holomorphic function $1/f$ and obtain that f must be constant in $D(z_0, r)$.

Theorem 5.4. Let A be a bounded domain and $f : \overline{A} \to \mathbb{C}$ a continuous function such that the restriction $f|_A$ is holomorphic. If $|f|$ attains its maximum in A, then f is constant on $\overline{A}$.

Proof. Set $M = \sup_{z \in \overline{A}} |f(z)|$. Since $\overline{A}$ is closed and bounded, it is compact, and so $M < \infty$ and $|f|$ attains its maximum M at some point $\hat{z} \in \overline{A}$. Assume now that there exists a point $z_0 \in A$ such that $|f(z_0)| = M$. Define

$$B = \{z \in A;\ |f(z)| = M\}.$$

Note that $B \neq \varnothing$ since $z_0 \in B$, and B is closed since $B = |f|_A|^{-1}(\{M\})$ is the inverse image of a closed set under the continuous map $|f|_A|$. We now argue that A is also open. Suppose $z \in B$ and take some $\epsilon > 0$ such that $D(z, \epsilon) \subseteq A$. Since $|f|$ attains a local maximum at z, we must have that $|f(w)| = M$ for all $w \in D(z, \epsilon)$ by the local maximum modulus principle. The latter means that $D(z, \epsilon) \subseteq B$, so B is open by arbitrariness of $z \in B$. Therefore, both B and $A \backslash B$ are open (relative in A), and since A is connected and B is nonempty, we must have $A = B$. Again since A is connected, $|f|$ can only be constant if f is constant. Finally, we see that f is also constant on $\overline{A}$ by continuity. □

Corollary 5.5. Under the same hypotheses as the theorem above, we have that

$$\max_{z\in\overline{A}} |f(z)| = \max_{z\in\partial A} |f(z)|.$$

Proof. We already know that the maximum is attained in $\overline{A}$. If it is attained in A, then $|f|$ is constant on $\overline{A}$ and there is nothing to prove. Otherwise, $|f|$ does not attain its maximum in A, so it must attain its maximum in ∂A, so we are done. $\square$

All assumptions in the global version of the maximum modulus theorem and its corollary are necessary:

- Take the unbounded domain $A = \{x + iy;\ x, y > 0\}$, then $f : \overline{A} \to \mathbb{C}$, $z \mapsto \exp(-iz^2)$ is holomorphic on A and fulfills $|f(z)| = 1$ for every $z \in \partial A$. However, $f(r(1+i)) = \exp(2r^2) \to \infty$ as $r \to \infty$.
- Take the disconnected set $A = D(0,1) \cup D(3,1)$ and define on $\overline{A} = \overline{D(0,1)} \cup \overline{D(3,1)}$ the function

$$f(z) = \begin{cases} 1, & z \in D(0,1), \\ 2, & z \in D(3,1), \end{cases}$$

  which is holomorphic in A. Then $|f|$ attains its maximum in $3 \in D(3,1)$, but f is not constant[7].

5.2 Harmonic Functions, Poisson Formula

We have already seens that the real and imaginary parts u and v of a holomorphic function $f = u + iv : A \to \mathbb{C}$ ($A \subseteq \mathbb{C}$ open) are necessarily harmonic due to Cauchy-Riemann equations, as long as they are twice continuously differentiable. In fact the latter part of the assumption is redundant, since we have already proved that f has arbitrarily many complex derivatives on A, indicating that u and v are smooth functions. We show that the opposite is also true.

Proposition 5.6. Let $A \subseteq \mathbb{C}$ be a domain and $u : A \to \mathbb{R}$ twice continuously differentiable and harmonic. Then,

(1) u is smooth (infinitely differentiable) and for $z_0 \in A$, there exists $r > 0$ and $f : D(z_0, r) \to \mathbb{C}$ holomorphic such that $u|_{D(z_0,r)} = \mathrm{Re}(f)$.

(2) If A is simply connected, then there is a holomorphic function $f : A \to \mathbb{C}$ such that $u = \mathrm{Re}(f)$.

Proof. We first prove (2). The function $g : A \to \mathbb{C}$, defined by

$$g(z) = \frac{\partial u}{\partial x}(x,y) - i\frac{\partial u}{\partial y}(x,y) =: U(x,y) + iV(x,y),$$

is holomorphic. Indeed, all partial derivatives $\frac{\partial U}{\partial x}$, $\frac{\partial U}{\partial y}$, $\frac{\partial V}{\partial x}$, and $\frac{\partial V}{\partial y}$ are continuous (since u is twice continuously differentiable), and we have that

$$\frac{\partial U}{\partial x}(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) = -\frac{\partial^2 u}{\partial y^2}(x,y) = \frac{\partial V}{\partial y}(x,y), \quad (x,y) \in A,$$

since that $\Delta u(x,y) = 0$. Moreover, by Schwarz' theorem, we have that

$$\frac{\partial U}{\partial y}(x,y) = \frac{\partial^2 u}{\partial y \partial x}(x,y) = \frac{\partial^2 u}{\partial x \partial y}(x,y) = -\frac{\partial V}{\partial x}(x,y), \quad (x,y) \in A.$$

Therefore, g is holomorphic. Since A is simply connected, Cauchy's integral theorem implies that g has a primitive f on A, so $f'(z) = g(z)$. Writing $f = \tilde{u} + i\tilde{v}$ then gives

$$f'(z) = \frac{\partial \tilde{u}}{\partial x}(x,y) - i\frac{\partial \tilde{u}}{\partial y}(x,y), \quad (x,y) \in A.$$

As a result, we have that $\frac{\partial}{\partial x}(u - \tilde{u}) = \frac{\partial}{\partial y}(u - \tilde{u}) = 0$ on A, and so $u - \tilde{u}$ is constant on A. By subtracting this constant, we will obtain $u = \mathrm{Re}(f)$ on A. Now we move on to prove (1). Since A is open, we can find $r > 0$ such that $D(z_0, r) \subseteq A$. Since $D(z_0, r)$ is simply connected, we restrict u to $D(z_0, r)$ to find by (2) a holomorphic function $f : D(z_0, r) \to \mathbb{C}$ with $u|_{D(z_0,r)} = \mathrm{Re}(f)$. Finally, since f has arbitrarily many derivatives, $u|_{D(z_0,r)}$ has arbitrarily many real derivatives in $D(z_0, r)$. Finally, by arbitrariness of $z_0 \in A$, u is smooth on all of A. $\square$

[7] The function f is however constant on the connected components of A

We say that the function v is a **harmonic conjugate** of u. So in other words, on a simply connected domain, every harmonic function has a harmonic conjugate. We now show that the mean value property (obtained for holomorphic functions) holds also for harmonic functions.

Proposition 5.7. Let $A \subseteq \mathbb{C}$ be open and $u : A \to \mathbb{C}$ be a harmonic function. Let $r > 0$ and $z_0 = (x_0, y_0) \in A$ such that $\overline{D(z_0, r)} \subseteq A$. Then

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + r \exp(it))dt.$$

Proof. We apply the previous proposition and find f holomorphic on $D(z_0, r)$ with $u = \mathrm{Re}(f)$. It then follows form the mean value property for holomorphic functions that

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r \exp(it))dt.$$

The result then follows by taking the real part and interchanging $\mathrm{Re}(\cdot)$ with the curve integral. $\square$

With the mean value property, we can obtain the maximum principle in exactly the same fashion as for the modulus of a holomorphic function. We formulate both the local and global versions in one theorem.

Theorem 5.8. Let $A \subseteq \mathbb{R}^2$ be open and $u : A \to \mathbb{R}$ harmonic. Then,

(1) Assume that for $r > 0$ and $(x_0, y_0) \in A$ with $D((x_0, y_0), r) \subseteq A$, we have either

$$u(x_0, y_0) \geq u(x, y), \qquad \text{for } (x, y) \in D((x_0, y_0), r),$$

or

$$u(x_0, y_0) \leq u(x, y), \qquad \text{for } (x, y) \in D((x_0, y_0), r).$$

Then u is constant on $D((x_0, y_0), r)$.

(2) Suppose additionally that A is connected and bounded, and u extends to a continuous function on $\overline{A}$. If u attains its minimum or maximum on A, then u is constant on $\overline{A}$. Moreover,

$$\max_{(x,y)\in\overline{A}} u(x, y) = \max_{(x,y)\in\partial A} u(x, y), \qquad \min_{(x,y)\in\overline{A}} u(x, y) = \min_{(x,y)\in\partial A} u(x, y).$$

Proof. For the statements about local or global maxima, we repeat the proof of previous theorems and corollaries for holomorphic functions, using the mean value property for harmonic functions instead. For the statements about minima, apply the respective statement about maxima to the harmonic function $-u$. $\square$

Consider now a bounded domain $A \subseteq \mathbb{R}^2$ and suppose we are given continuous functions $g : A \to \mathbb{R}$ and $\xi : \partial A \to \mathbb{R}$. We say that a function $u : \overline{A} \to \mathbb{R}$ that is twice continuously differentiable in the real sense in A fulfills the **Poisson equation** with boundary condition ξ, if

$$\begin{aligned} \Delta u(x, y) &= g(x, y), \qquad &&\text{for } (x, y) \in A, \\ u(x, y) &= \xi(x, y), \qquad &&\text{for } (x, y) \in \partial A. \end{aligned}$$

The special case where $g \equiv 0$ is called the **Dirichlet problem**. Proving the existence of a solution to these problems is non-trivial and depends on the regularity of the boundary (this involves methods form the theory of partial differential equations, such as Perron's method, or alternatively can be studied using Brownian motion). It is however very easy to show that a solution to this problem must be unique.

Proposition 5.9. The solution u to a Poisson equation is unique.

Proof. Let u_1 and u_2 be two solutions to a Poisson equation. The $u = u_2 - u_1$ is a harmonic function and $u|_{\partial A} = 0$. Applying the maximum principle, we can obtain that

$$0 = \min_{(x,y)\in\partial A} u(x, y) = \min_{(x,y)\in\overline{A}} u(x, y) \leq \max_{(x,y)\in\overline{A}} u(x, y) = \max_{(x,y)\in\partial A} u(x, y) = 0,$$

which implies $u \equiv 0$ on $\overline{A}$. $\square$

In the case of $A = D(0,r)$, one can in fact give an explicit solution to the Dirichlet problem relying on Cauchy's integral formula.

Theorem 5.10. Let $r > 0$ and $u : \overline{D(0,r)} \to \mathbb{R}$ a continuous function harmonic in $D(0,r)$. For $0 < \rho < r$ and $\theta \in (-\pi, \pi]$, we have **Poisson's formula**

$$u(\rho \exp(i\theta)) = \frac{r^2 - \rho^2}{2\pi} \int_0^{2\pi} \frac{u(r\exp(it))}{r^2 - 2r\rho\cos(\theta - t) + \rho^2} dt,$$

or equivalently

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(r\exp(it)) \cdot \frac{r^2 - |z|^2}{|r\exp(it) - z|^2} dt.$$

Proof. Since $D(0,r)$ is simply connected and u is harmonic on $D(0,r)$, we can find a holomorphic function $f : D(0,r) \to \mathbb{C}$ with $u = \mathrm{Re}(f)$. Using Cauchy's integral formula for $0 < s < r$, we obtain

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(0,s)} \frac{f(w)}{w - z} dw, \qquad |z| < s.$$

We set $\tilde{z} = s^2/\overline{z}$ (the **reflection** of z on the circle $\partial D(0,s)$) and obtain that

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(0,s)} f(w) \left(\frac{1}{w - z} - \frac{1}{w - \tilde{z}} \right) dw, \qquad |z| < s,$$

since the function $w \mapsto \frac{f(w)}{w - \tilde{z}}$ is holomorphic in $D(0,s')$ for some $s' > s$. We can then rewrite

$$\frac{1}{w-z} - \frac{1}{w-\tilde{z}} = \frac{1}{w-z} - \frac{1}{w - |w|^2/\overline{z}} = \frac{1}{w-z} - \frac{\overline{z}}{w\overline{z} - w\overline{w}} = \frac{|w|^2 - |z|^2}{w|w-z|^2}.$$

Hence,

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(0,s)} \frac{f(w)(|w|^2 - |z|^2)}{w|w-z|^2} dw.$$

Inserting the standard parametrization $\gamma_{\partial D(0,s)} : [0, 2\pi] \to \mathbb{C}$, $t \mapsto s\exp(it)$ and writing $z = \rho\exp(i\theta)$ for $\rho = |z| \in [0, s)$, we obtain that

$$f(\rho\exp(i\theta)) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(s\exp(it))(s^2 - \rho^2)}{|s\exp(it) - \rho\exp(i\theta)|^2} dt.$$

Taking the real part, we obtain that

$$u(\rho\exp(i\theta)) = \frac{1}{2\pi} \int_0^{2\pi} \frac{u(s\exp(it))(s^2 - \rho^2)}{s^2 + \rho^2 - 2s\rho\cos(\theta - t)} dt.$$

Keep ρ and $t \in [0, 2\pi]$ fixed, then

$$\frac{u(s\exp(it))(s^2 - \rho^2)}{s^2 + \rho^2 - 2s\rho\cos(\theta - t)} \xrightarrow{u.} \frac{u(r\exp(it))(r^2 - \rho^2)}{r^2 + \rho^2 - 2r\rho\cos(\theta - t)} \quad \text{as } s \to r,$$

since the function above is continuous on a compact set. Therefore, it holds that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{u(s\exp(it))(s^2 - \rho^2)}{s^2 + \rho^2 - 2s\rho\cos(\theta - t)} dt \longrightarrow \frac{1}{2\pi} \int_0^{2\pi} \frac{u(r\exp(it))(r^2 - \rho^2)}{r^2 + \rho^2 - 2r\rho\cos(\theta - t)} dt,$$

which shows the claim. □

The previous theorem gives a formula to show how u is given in $D(0,r)$ if u solves the Dirichlet problem. Conversely, we can simply argue that inserting the boundary values ξ on $\partial D(0,r)$, we do in fact obtain a solution of the Dirichlet problem. See lecture notes for the explicit proof.

11/14 Lecture

6 Power Series and Taylor's Theorem

6.1 Sequences and Series of Holomorphic Functions

Definition 6.1. Let $A \subseteq \mathbb{C}$ and $(f_n)_{n\in\mathbb{N}}$ a sequence of functions $f_n : A \to \mathbb{C}$.

(1) We say that $(f_n)_{n\in\mathbb{N}}$ **converges uniformly on** A to some function $f : A \to \mathbb{C}$, if for every $\epsilon > 0$, there is an $N = N(\epsilon) \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \epsilon, \qquad \text{for all } n \in \mathbb{N},\ z \in A.$$

In this case, we write $f_n \overset{\text{u}}{\to} f$ (on A) as $n \to \infty$.

(2) Similarly, the series $\sum_{n=1}^{\infty} f_n(z)$ is said to **converge uniformly on** A if the sequence of functions $(S_n)_{n\in\mathbb{N}}$ of functions $S_n : A \to \mathbb{C}$, defined by

$$S_n(z) = \sum_{\nu=1}^{n} f_\nu(z),$$

converges uniformly.

Note that uniform convergence implies pointwise convergence $\lim_{n\to\infty} f_n(z) = f(z)$, but not vice versa.

Theorem 6.2. Let $A \subseteq \mathbb{C}$. Consider the sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n : A \to \mathbb{C}$.

(1) The sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly if and only if for every $\epsilon > 0$, there is an $N = N(\epsilon) \in \mathbb{N}$ such that

$$|f_n(z) - f_m(z)| < \epsilon, \qquad \text{for all } n, m \geq M,\ z \in A.$$

(2) The series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly if and only if for every $\epsilon > 0$, there is an $N = N(\epsilon) \in \mathbb{N}$ such that

$$\left| \sum_{\nu=n+1}^{n+p} f_\nu(z) \right| < \epsilon, \qquad \text{for all } n \geq N,\ p \in \mathbb{N},\ z \in A.$$

Proof. (1) ($\Longrightarrow$) Assume that $f_n \overset{\text{u}}{\to} f$ on A. For $\epsilon > 0$, let $N = N(\epsilon) \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \epsilon/2$ for all $n \geq N$ and all $z \in A$. Then, for $z \in A$,

$$|f_n(z) - f_m(z)| \leq |f_n(z) - f(z)| + |f_m(z) - f(z)| < \epsilon, \qquad \forall n, m \geq N.$$

($\Longleftarrow$) Let $f(z) = \lim_{n\to\infty} f_n(z)$ (which exists since for every fixed $z \in A$, $(f_n(z))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$, and so converges). For $\epsilon > 0$, choose $N = N(\epsilon) \in \mathbb{N}$ such that

$$|f_n(z) - f_m(z)| < \frac{\epsilon}{2}, \qquad \forall n, m \geq N.$$

Now for $z \in A$, we find $M = M(\epsilon, z) \in \mathbb{N}$ large enough (depending also on z) such that

$$|f_m(z) - f(z)| < \frac{\epsilon}{2}, \qquad \forall m \geq M.$$

Therefore, we find that for every $n \geq N$ and $z \in A$, we have that

$$|f_n(z) - f(z)| \geq |f_n(z) - f_m(z)| + |f_m(z) - f(z)| < \epsilon,$$

where in the intermediate set, it suffices to choose $m \geq \max\{M(z, \epsilon), N\}$.

(2) This follows from (1) directly, applied to the partial sums. □

Lemma 6.3. Let $A \subseteq \mathbb{C}$. Consider a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions $f_n : A \to \mathbb{C}$. Then,

(1) Assume that $f_n \overset{\text{u}}{\to} f$ on A as $n \to \infty$ for some function $f : A \to \mathbb{C}$. Then f is continuous.

(2) If $\sum_{n=1}^{\infty} f_n(z) =: g(z)$ converges uniformly on A, then g is continuous.

Proof. (1) Consider $z_0 \in A$. Let $\epsilon > 0$ and choose first $N \in \mathbb{N}$, and then $\delta > 0$, such that $|f_N(z) - f_N(z_0)| < \epsilon/3$ for $z \in D(z_0, \delta) \cap A$ (by continuity), and $|f_N(z_0) - f(z_0)| < \epsilon/3$, $|f_N(z) - f(z)| < \epsilon/3$ for $z \in A$ (by uniform convergence). This will give

$$|f(z) - f(z_0)| \leq |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| + |f_N(z_0) - f(z_0)| < \epsilon,$$

which holds when $z \in D(z_0, \delta) \cap A$.

(2) This follows from (1) directly, applied to the partial sums. ☐

Theorem 6.4 (Weierstrass M test)**.** Let $A \to \mathbb{C}$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $f_n : A \to \mathbb{C}$. Suppose that there is a sequence of non-negative real numbers $(M_n)_{n\in\mathbb{N}}$ with

(1) $|f_n(z)| \leq M_n$ for all $z \in A$, $n \in \mathbb{N}$, and

(2) $\sum_{n=1}^{\infty} M_n$ converges.

Then the series $\sum_{n=1}^{\infty} f_n(z)$ converges **uniformly absolutely** on A, meaning that $\sum_{n=1}^{\infty} |f_n(z)|$ converges uniformly on A. Moreover, $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on A.[8]

Proof. Since $\sum_{n=1}^{\infty} M_n$ converges, the sequence of partial sums is a Cauchy sequence, so for $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$, such that $\sum_{\nu=n+1}^{n+p} M_\nu \leq \epsilon$ for $n \geq N$, $p \in \mathbb{N}$. Therefore, we have that

$$\left|\sum_{\nu=n+1}^{n+p} f_\nu(z)\right| \leq \sum_{\nu=n+1}^{n+p} |f_\nu(z)| \leq \sum_{\nu=n+1}^{n+p} M_\nu < \epsilon, \qquad \forall n \geq N,\ p \in \mathbb{N},\ z \in A.$$

Both statements therefore follow by applying the Cauchy criterion. ☐

Example 6.5. For fixed $0 \leq q < 1$, $\sum_{n=0}^{\infty} z^n$ converges uniformly absolutely on $\overline{D(0,q)}$, since $|z^n| \leq q^n$ for $|z| \leq q$ and $\sum_{n=0}^{\infty} q^n < \infty$. With $M_n := q^n \geq 0$, the previous theorem thus applies.

Proposition 6.6. Let $A \subseteq \mathbb{C}$ be open and $\gamma : [a,b] \to \mathbb{C}$, $a < b$ be a piecewise continuously differentiable curve with $\gamma([a,b]) \subseteq A$. Also let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous function $f_n : A \to \mathbb{C}$, with $f_n \overset{\text{u}}{\to} f$ on every compact subset $K \subseteq A$ as $n \to \infty$ for some $f : A \to \mathbb{C}$. Then,

$$\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma f(z)dz.$$

Moreover, if $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on every compact subset $K \subseteq A$ as $n \to \infty$, we have that

$$\sum_{n=1}^{\infty} \int_\gamma f_n(z)dz = \int_\gamma \left(\sum_{n=1}^{\infty} f_n(z)\right) dz.$$

Proof. First we note that f is continuous on A. Also note that for every $z \in A$, there is $\epsilon > 0$ with $\overline{D(z,\epsilon)} \subseteq A$. Now we assume that $l(\gamma) > 0$, since otherwise $l(\gamma) = 0$ and thus all integrals vanish, so there is nothing to prove. Since γ is continuous and $\gamma([a,b]) \subseteq A$ is compact, by uniform convergence, we have that for a given $\epsilon > 0$ there exists $N \in \mathbb{N}$, such that

$$|f_n(z) - f(z)| < \frac{\epsilon}{l(\gamma)}, \qquad \forall n \geq N,\ z \in \gamma([a,b]).$$

Now we use the standard bound to infer that

$$\left|\int_\gamma f_n(z)dz - \int_\gamma f(z)dz\right| \leq \sup_{z\in\gamma([a,b])} |f_n(z) - f(z)| \cdot l(\gamma) \leq \epsilon, \qquad \forall n \in \mathbb{N}.$$

This completes the proof of the first statement. The proof of the second statement follows directly by applying the same reasoning to the partial sums. ☐

[8] Every series that converges uniformly absolutely on some A also converges uniformly on A, as can be seen by Cauchy's criterion.

Theorem 6.7 (Weierstrass approximation theorem)**.** Let $A \subseteq \mathbb{C}$ be open and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of holomorphic functions $f_n : A \to \mathbb{C}$.

(1) Assume that $f_n \overset{\text{u}}{\to} f$ on every compact $K \subseteq A$ for some $f : A \to \mathbb{C}$. Then f is holomorphic and it holds that

$$\begin{aligned} &f_n'(z) \to f'(z), \quad \text{for } z \in A, \\ &f_n' \overset{\text{u}}{\to} f', \quad \text{on every compact set } K \subseteq A, \text{ as } n \to \infty. \end{aligned}$$

(2) If $g(z) = \sum_{n=1}^{\infty} f_n(z)$ converges uniformly on every compact $K \subseteq A$, then g is holomorphic and

$$g'(z) = \sum_{n=1}^{\infty} f_n'(z), \qquad \text{for } z \in A,$$

and the convergence is uniform on every compact $K \subseteq A$.

Proof. (1) To show that f is holomorphic, consider $z_0 \in A$ and take $\epsilon > 0$ such that $\overline{D(z_0, \epsilon)} \subseteq A$. We know that f must be continuous on $\overline{D(z_0, \epsilon)}$. Consider a triangle $\triangle(z_1, z_2, z_3) \subseteq D(z_0, \epsilon)$. By the previous proposition and Cauchy's integral theorem for star-shaped domains, we thus obtain

$$\int_{\gamma_{\partial\triangle(z_1,z_2,z_3)}} f(z)dz = \lim_{n\to\infty} \int_{\gamma_{\partial\triangle(z_1,z_2,z_3)}} f_n(z)dz = 0.$$

By Morera's theorem, we see that f is holomorphic on $D(z_0, \epsilon)$, and by arbitrariness of z_0, we have that f is holomorphic on A. Now let $K \subseteq A$ be compact and let $r > 0$ be such that $d(K, \partial A) > r$. We can cover K with finitely many open disks $D(z_j, r)$, $j = 1, \cdots, M$ and choose $\rho > r$ so that $\overline{D(z_j, \rho)} \subseteq A$ for every $j = 1, \cdots, M$. Now let $z \in K$, then $z \in D(z_{j_\star}, r)$ for some $j_\star \in \{1, \cdots, M\}$. We apply Cauchy's integral formula for derivatives to see that

$$f_n'(z) = \frac{1}{2\pi i} \int_{\partial D(z_{j_\star},\rho)} \frac{f_n(w)}{(w-z)^2} dw, \quad f'(z) = \frac{1}{2\pi i} \int_{\partial D(z_{j_\star},\rho)} \frac{f(w)}{(w-z)^2} dw.$$

By assumption, f_n converges uniformly to f on every compact subset of A, in particular on all $\partial D(z_j, \rho)$, $j = 1, \cdots, M$, so we see that for $\epsilon > 0$ there exists an $N = N(\epsilon) \in \mathbb{N}$, such that $|f_n(w) - f(w)| < \epsilon$ for all $n \geq N$ and $w \in \partial D(z_j, \rho)$, $j = 1, \cdots, M$. Now note that since $z \in D(z_{j_\star}, r)$ and $w \in \partial D(z_{j_\star}, \rho)$, we find that $|w - z| \geq \rho - r$ by the inverse triangle inequality. Therefore we obtain that

$$|f_n'(z) - f'(z)| = \left| \frac{1}{2\pi i} \int_{\partial D(z_{j_\star},\rho)} \frac{f_n(w) - f(w)}{(w-z)^2} dw \right| \leq \frac{1}{2\pi} \frac{\epsilon l(\gamma_{\partial D(z_{j_\star},\rho)})}{(\rho - r)^2} = \frac{\epsilon\rho}{(\rho - r)^2}, \quad \forall n \geq N.$$

Note that ρ and r are independent of the choice of $z \in K$, so we obtain the desired uniform convergence on K. Note that this implies pointwise convergence on K, and the pointwise convergence on A follows since every $z \in A$ is contained in some $\overline{D(z, \epsilon)}$.

□

6.2 Power Series

Definition 6.8. An infinite series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where $a_n \in \mathbb{C}$, is called a **power series** with coefficients $(a_n)_{n\in\mathbb{N}}$ around $z_0 \in \mathbb{C}$.

Lemma 6.9. If the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ is convergent for some $z = z_1 \in \mathbb{C}$ with $|z_1 - z_0| = r$, then it converges for every $z \in D(z_0, r)$. Moreover, the power series converges uniformly absolutely in $\overline{D(z_0, \rho)}$ for every $\rho \in [0, r)$.

Proof. There is nothing to show for $r = 0$, so assume $r > 0$. For $z = z_1$, the series is convergent, which implies in particular that its terms are bounded. Hence, there is a $c > 0$ with $|a_n|r^n = |a_n(z_1 - z_0)^n| \leq c$ for all $n \in \mathbb{N}$, or in other words,

$$|a_n| \leq \frac{c}{r^n}, \qquad \forall n \in \mathbb{N}.$$

Now for $|z - z_0| \leq \rho < r$, we see that

$$|a_n(z - z_0)^n| = |a_n||z - z_0|^n \leq c \left(\frac{\rho}{r}\right)^n.$$

Now the claim follows by the Weierstrass M test for $M_n = (\rho/r)^n$, since $\rho/r \in [0, 1)$. ☐

Theorem 6.10. For every given power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, there exists a unique $R \in [0, \infty) \cup \{\infty\}$, such that

(1) The series converges uniformly absolutely on every $\overline{D(z_0, \rho)}$ if $0 \leq \rho < R$.

(2) For $|z - z_0| > R$, the series diverges.

(3) The **formula of Cauchy-Hadamard** holds:

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}},$$

with the convention that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

(4) If the limit

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$$

exists, then it must be equal to R.

The number R is called the **radius of convergence** of the power series.

Proof. The proof is the same as in real analysis, and will be ignored here. For details, check the lecture notes. ☐

We say that the set $\{z \in \mathbb{C};\ |z - z_0| = R\}$ is the **circle of convergence** of the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$. We will now show that a power series is holomorphic inside its circle of convergence.

Theorem 6.11. A power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ defines a holomorphic function

$$f : D(z_0, R) \to \mathbb{C}, \qquad z \mapsto \sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

where R is the radius of convergence. Moreover, we have that

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1},$$

and the power series on the right-hand side of this equation has again R as its radius of convergence. The coefficients a_n, $n \in \mathbb{N}$, are given by the formula

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_{\partial D(z_0, r)} \frac{f(w)}{(w - z_0)^{n+1}} dw, \qquad \text{for any } r \in (0, R).$$

Proof. Assume that $K \subseteq D(z_0, R)$ is compact, then $K \subseteq \overline{D(0, \rho)}$ for some $\rho < R$, and thus by the previous theorem, the series converges uniformly on K. We now show that the power series representing f' has R as its radius of convergence. Indeed, we write for $z \neq z_0$ that

$$\sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1} = \frac{1}{z - z_0} \sum_{n=0}^{\infty} n a_n (z - z_0)^n.$$

Note that the multiplication by $\frac{1}{z-z_0}$ does not change whether the power series converges or diverges, so we can apply the Cauchy-Hadamard formula

$$\frac{1}{\limsup_{n\to\infty} |na_n|^{\frac{1}{n}}} = \frac{1}{\limsup_{n\to\infty} n^{\frac{1}{n}} |a_n|^{\frac{1}{n}}} = R.$$

Finally, by iterating this procedure, we see that for $k \in \mathbb{N}$, we have that

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z-z_0)^{n-k}.$$

Inserting $z = z_0$ in this formula shows that $f^{(k)}(z_0) = k!a_k$, which yields the desired expression for a_k in view of the generalized version of Cauchy's integral formula. □

Remark 6.12. (1) The theorem above shows that if a function f is given by a power series, we must have that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$$

(2) It is also clear that a function f given by a power series must have a primitive (since it is a holomorphic function on the star-shaped domain $D(z_0, R)$), and the theorem above can be used to find a primitive. Indeed, we readily verify that

$$F : D(z_0, R) \to \mathbb{C}, \qquad z \mapsto \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1}$$

has f as its derivative (and since taking the derivative does not change the radius of convergence, F has the same radius of convergence as f).

6.3 Taylor's Theorem

In the previous section we saw that a function given by a convergent power series is holomorphic and is given by the Taylor's series. In this section we will show the converse: Ant holomorphic function can be locally developed into a Taylor series.

Theorem 6.13 (Taylor's theorem)**.** Let $z_0 \in \mathbb{C}$, $r > 0$ and $f : D(z_0, r) \to \mathbb{C}$ holomorphic. Then for every $z \in D(z_0, r)$, we have that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$$

In particular, the series on the right-hand side of the equation above converges.

Proof. Let $|z| < r$ and define $h(z) = f(z_0 + z)$, then h is holomorphic on $D(0, r)$. For $\rho \in (0, r)$, the Cauchy integral formula yields that

$$h(z) = \frac{1}{2\pi i} \int_{\partial D(0,\rho)} \frac{h(w)}{w-z} dw, \qquad z \in D(0, \rho).$$

Note that for $\xi \neq 1$ and $n \in \mathbb{N}$, we have that

$$1 + \xi + \cdots + \xi^{n-1} = \frac{1-\xi^n}{1-\xi} \qquad \Longrightarrow \qquad \frac{1}{1-\xi} = 1 + \xi + \cdots + \xi^{n-1} + \frac{\xi^n}{1-\xi}.$$

We use this for $\xi = z/w$ with $w \in \partial D(0, \rho)$ and $z \in D(0, \rho)$ (so that $|\xi| < 1$), giving that

$$\frac{1}{w-z} = \frac{1}{w} \cdot \frac{1}{1-\frac{z}{w}} = \frac{1}{w} \cdot \left(1 + \frac{z}{w} + \cdots + \left(\frac{z}{w}\right)^{n-1} + \frac{\left(\frac{z}{w}\right)^n}{1-\frac{z}{w}}\right) = \frac{1}{w} + \frac{z}{w^2} + \cdots + \frac{z^{n-1}}{w^n} + \frac{z^n}{w^n}\frac{1}{w-z}.$$

Therefore, we find that

$$h(z) = \sum_{\nu=0}^{n-1} z^\nu \left(\frac{1}{2\pi i} \int_{\partial D(0,\rho)} \frac{h(w)}{w^{\nu+1}} dw \right) + \frac{z^n}{2\pi i} \int_{\partial D(0,\rho)} \frac{h(w)}{w^n(w-z)} dw.$$

We see that by the generalized Cauchy's integral formula,

$$\frac{1}{2\pi i} \int_{\partial D(0,\rho)} \frac{h(w)}{w^{\nu+1}} = \frac{h^{(\nu)}(0)}{\nu!}, \qquad \nu \in \{0, \cdots, n-1\}.$$

In other words,

$$h(z) = \sum_{\nu=0}^{n-1} \frac{h^{(\nu)}(0)}{\nu!} z^\nu + \frac{z^n}{2\pi i} \int_{\partial D(0,\rho)} \frac{h(w)}{w^n(w-z)} dw.$$

Note that for $w \in \partial D(0,\rho)$, we have for some $\tilde{\rho} \in (0,\rho)$ with $|z| \le \tilde{\rho}$ that $|w-z| \ge \rho - \tilde{\rho}$. Hence, by setting $M = \max_{w \in \partial D(0,\rho)} |h(w)|$, we find that

$$\left| \frac{z^n}{2\pi i} \int_{\partial D(0,\rho)} \frac{h(w)}{w^n(w-z)} dw \right| \le \frac{\tilde{\rho}^n}{2\pi} \frac{M}{\rho^n(\rho-\tilde{\rho})} \cdot l(\gamma_{\partial D(0,\rho)}) = \left(\frac{\tilde{\rho}}{\rho}\right)^n \frac{M\rho}{\rho-\tilde{\rho}} \to 0 \qquad \text{as } n \to \infty.$$

Therefore, we find that

$$h(z) = \sum_{\nu=0}^{\infty} \frac{h^{(\nu)}(0)}{\nu!} z^\nu, \qquad \forall z \in \mathbb{C},\ |z| \le \tilde{\rho}.$$

By arbitratiness of $0 < \tilde{\rho} < \rho < r$, we thus conclude our proof. □

Corollary 6.14. Let $A \subseteq \mathbb{C}$ be open and $f : A \to \mathbb{C}$ a mapping. Then the following are equivalent.

(1) f is holomorphic on A.

(2) f is analytic on A in the sense that for every $z_0 \in A$, there is $\epsilon > 0$ with $D(z_0,\epsilon) \subseteq A$ and a power series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n,$$

converging on $D(z_0,\epsilon)$ and being equal to $f|_{D(z_0,\epsilon)}$.

Proof. Assume that f is holomorphic. For $z_0 \in A$, there is $\epsilon > 0$ with $D(z_0,\epsilon) \subseteq A$. Then (2) follows from Taylor's theorem. Conversely suppose that the statements in (2) holds. Then for a fixed $z_0 \in A$ and $\epsilon > 0$ with $D(z_0,\epsilon) \subseteq A$, $f|_{D(z_0,\epsilon)}$ by Theorem 6.11. The proof is thus completed by arbitrariness of z_0. □

This corollary justifies that the terms "holomorphic" and "analytic" may be used interchangeably. Every holomorphic function can therefore locally be expressed as a Taylor series.

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Example 6.15. (1) The Taylor series of $\exp(z)$, $\sin(z)$, and $\cos(z)$ are as in the real case and valid for all $z \in \mathbb{C}$.

(2) The principal value of the logarithm Log on $\mathbb{C}_-$ has the Taylor series

$$\operatorname{Log}(z) = \operatorname{Log}(z_0) + \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{z_0^\nu \nu} (z-z_0)^\nu, \qquad \forall z \in D(z_0,r) \subseteq \mathbb{C}_-.$$

Note that by the Cauchy-Hadamard formula, the radius of convergence of the power series on the right-hand side is $|z_0|$. If however $\operatorname{Re}(z_0) < 0$, then $D(z_0,|z_0|) \not\subseteq \mathbb{C}_-$. So the radius of convergence of the Taylor series representing a holomorphic function need not be contained in the domain of this function.

We give some computation rules for power series.

- Identity of power series: Two convergent power series that represent the same function has all their corresponding coefficients the same.
- Sum of power series: The sum of two convergent power series is the power series with the coefficients as the sum of their corresponding coefficients. The radius of convergence of the sum is the minimum between the original radii of convergence.
- Product of power series: If two power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ have radii of convergence $R_1, R_2 > 0$, then one has the **Cauchy product formula**

$$\left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=0}^{\infty} c_n z^n, \qquad \forall |z| < \min\{R_1, R_2\},$$

  where

$$c_n = \sum_{\nu=0}^{n} a_\nu b_{n-\nu}.$$

  The formula follows by using that $(fg)^{(n)}(0) = \sum_{\nu=0}^{n} \binom{n}{\nu} f^{(\nu)}(0) g^{(n-\nu)}(0)$.

- Multiplicative inversion of power series: Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$ and $a_0 \neq 0$. Then there exists $\epsilon > 0$, such that $f(z) \neq 0$ for all $z \in D(0, \epsilon)$, on which $g = 1/f$ is holomorphic. Developing g into a Taylor series $g(z) = \sum_{n=0}^{\infty} b_n z^n$ for $z \in D(0, \epsilon)$, the coefficients must fulfill

$$\sum_{\nu=0}^{n} a_\nu b_{n-\nu} = \begin{cases} 1, & n = 0, \\ 0, & n \geq 1. \end{cases}$$

- Sequences of power series: Assume that the power series

$$f_j(z) = \sum_{n=0}^{\infty} c_{jn} z^n, \qquad j \in \mathbb{N},$$

  all converge in $D(0, R)$, $R > 0$ and assume that $|f_n(z)| \leq M_n$ for some sequence $(M_n)_{n \geq 0}$ of non-negative real numbers such that $\sum_{n=0}^{\infty} M_n$ converges. Then the function $F = \sum_{j=0}^{\infty} f_j$ is holomorphic on $D(0, R)$, and we have that

$$F(z) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} c_{jn}\right) z^n.$$

  Indeed, this follows from the Weierstrass M test and Weierstrass approximation theorem.

- Composition of power series: Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ (*i.e.*, $g(0) = 0$) with positive radius of convergence. Then $f \circ g$ has a positive radius of convergence and for $f(g(z)) = \sum_{n=0}^{\infty} c_n z^n$, we have that

$$\begin{aligned} c_0 &= f(g(0)) = f(0) = a_0, \\ c_1 &= f'(g(0))g'(0) = a_1 b_1, \\ c_2 &= \frac{f''(g(0))g'(0)^2 + f'(g(0))g''(0)}{2} = a_2 b_1^2 + a_1 b_2, \\ &\vdots \end{aligned}$$

6.4 Mapping Properties of Holomorphic Functions

Definition 6.16. Let $B \subseteq A \subseteq \mathbb{C}$. A point $z \in A$ is called an **accumulation point of** B **in** A if

$$\forall \epsilon > 0, \ \exists w \in B, \text{ such that } 0 < |w - z| < \epsilon.$$

Proposition 6.17 (The isolation of zeros of holomorphic functions)**.** Let $A \subseteq \mathbb{C}$ be a domain and $f : A \to \mathbb{C}$ holomorphic and not identically equal to zero. Then the set of zeros $N(f) = \{z \in A;\ f(z) = 0\}$ does not have an accumulation point in A.

Proof. Assume that there is an accumulation point of $z_0 \in A$ in $N(f)$, and therefore a sequence $(z_n)_{n\in\mathbb{N}} \subseteq N(f)\setminus\{z_0\}$ with $z_n \to z_0$ as $n \to \infty$. There exists $r > 0$ such that

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu, \qquad \forall z \in D(0, r).$$

Since f is continuous, we must have $a_0 = f(z_0) = \lim_{n\to\infty} f(z_n) = 0$. This argument can now be repeated for the power series

$$\frac{f(z)}{z - z_0} = \sum_{\nu=0}^{\infty} a_{\nu+1}(z - z_0)^\nu,$$

which implies that $a_1 = 0$ as well. By iteration, we find that $a_\nu = 0$ for all $\nu \in \mathbb{N}$, thus $f \equiv 0$ on $D(0, r)$. Now we consider the set

$$U = \{z \in A;\ z \text{ is an accumulation point of } N(f) \subseteq A\}.$$

By assumption, $z_0 \in U$, so $U \neq \varnothing$. Furthermore, U is open since all points within some disk around an accumulation point are also accumulation points as we have just seen. We now show that U is also closed (relative to A). Indeed, let $z_1 \in A\setminus U$, then there exists some disk $D(z_1, \epsilon) \subseteq A\setminus U$. This is because otherwise, for every $\epsilon > 0$, there exists $w \in \dot{D}(z_1, \epsilon) \cap U$, and since w is an accumulation point of $N(f)$, there exists $\xi \in N(f)$ such that $0 < |w - \xi| < \min\{|w - z_1|, \epsilon - |w - z_1|\}$. This implies that $0 < |z_1 - \xi| < \epsilon$, which means $\xi \in D(z_1, \epsilon) \subseteq A\setminus U$, leading to a contradiction. We thus see that $A\setminus U$ is open, and so U is closed (relative to A). Since A is connected, we see that $U = A$. But by continuity, this would imply that $f \equiv 0$ on A, leading to a contradiction. Therefore, the initial assumption must have been false, so $N(f)$ cannot have an accumulation point in A. $\square$

Note that while the set of zeros does not have any accumulation point in A, it may of course have accumulation points in $\mathbb{C}$. For instance, the mapping $f : \mathbb{C}\setminus\{0\} \to \mathbb{C}$, $z \mapsto \sin\left(\frac{2\pi}{z}\right)$ is holomorphic with $N(f) = \left\{\frac{1}{n};\ n \in \mathbb{N}\right\}$, having the accumulation point $0 \in \mathbb{C}$ but $0 \notin \mathbb{C}\setminus\{0\}$.

Theorem 6.18 (Identity theorem for holomorphic functions)**.** Let $A \subseteq \mathbb{C}$ be a domain and $f, g : A \to \mathbb{C}$ be two holomorphic functions. Then the following are equivalent:

(1) $f = g$.

(2) The coincidence set $\{z \in A;\ f(z) = g(z)\}$ has an accumulation point in A.

(3) There exists a point $z_0 \in A$ such that $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all $n \in \mathbb{N}$.

Proof. The equivalence of (1) and (2) follows immediately from the isolation of zeros of holomorphic functions, applied to the function $f - g$. Since (1) trivially implies (3), it suffices to show that (3) implies (2). By Taylor's theorem, the functions f and g agree on $D(z_0, r)$ for some $r > 0$, and (2) follows. $\square$

Remark 6.19. (1) The identity theorem asserts that for a domain $A \subseteq \mathbb{C}$, the values of a holomorphic function $f : A \to \mathbb{C}$ is already determined by the values on a *very small* subset. There is a strong "global" rigidity in the values of a complex function. The situationin $\mathbb{R}$ is of course completely different: The mere existence of smooth functions with compact support (*i.e.*, vanishing outside a compact subset) shows the failure of the identity theorem.

(2) It is crucial for the validity of the identity theorem that A is connected. Indeed, if A is only open, say $A = D(0, 1) \cup D(3, 1)$, then the mapping $f : z \mapsto 1$ on $D(0, 1)$ and $z \mapsto 0$ on $D(3, 1)$ is holomorphic and identical to 0 on $D(3, 1)$, but not identical to 0 on A.

(3) Holomorphic extensions are unique in the following sense: Let $A \subseteq \mathbb{C}$ be a domain and $M \subseteq A$ be a set that contains at least one accumulation point in A. Let $f : M \to \mathbb{C}$ be map. If a holomorphic function $\tilde{f} : A \to \mathbb{C}$ exists with $\tilde{f}|_M = f$, then it is unique with this property. In particular, the real functions exp, sin, and cos on $\mathbb{R}$ have unique extensions to $\mathbb{C}$.

Theorem 6.20 (Open mapping theorem)**.** Let $A \subseteq \mathbb{C}$ be a domain and $f : A \to \mathbb{C}$ be a non-constant holomorphic function. Then $f(A)$ is also a domain.

Proof. Since f is continuous and A is connected, $f(A)$ is connected as well. We need to show that $f(A)$ is also open. Let $z_0 \in A$ and consider $w_0 = f(z_0)$. Choose $r > 0$ small such that $\overline{D(z_0, r)} \subseteq A$ and f does not attain the value w_0 on $\partial D(z_0, r)$. This is possible since the zeros of the mapping $z \mapsto f(z) - w_0$ are isolated and note that z_0 is a zero of this mapping. Therefore, we have that

$$\delta = \min_{z \in \partial D(z_0,r)} |f(z) - w_0| > 0.$$

Thus for $|w - w_0| < \delta/2$ and $z \in \partial D(z_0, r)$, we have that

$$|f(z) - w| \geq ||f(z) - w_0| - |w - w_0|| > \delta - \frac{\delta}{2} = \frac{\delta}{2}.$$

Since $|f(z_0) - w| < \delta/2$, we see that the mapping $z \mapsto f(z) - w$ must have a zero in $D(z_0, r)$. This in turn implies that f attains w in $D(z_0, r)$, and therefore $D(w_0, \delta/2) \subseteq f(A)$. $\square$

The open mapping theorem gives an alternative proof for the fact that a holomorphic function $f : A \to \mathbb{C}$ on a domain A must be constant if one of $\mathrm{Re}(f)$, $\mathrm{Im}(f)$, or $|f|$ is constant. Indeed, this would imply that $f(A)$ is not a domain, and so f must be constant.

7 Singularities and Laurent Decomposition

7.1 Singularities

Definition 7.1. Let $A \subseteq \mathbb{C}$ open and $f : A \to \mathbb{C}$ holomorphic. Assume that $z_0 \in \mathbb{C} \backslash A$, but $\dot{D}(z_0, r) \subseteq A$ for some $r > 0$. Then z_0 is called an *(isolated) singularity* of f.

In what follows, we consider only isolated singularities and we will say "singularity" instead of "isolated singularity". We will see three types of singularities: removable singularities, poles, and essential singularities.

Definition 7.2. Let $A \subseteq \mathbb{C}$ open, $f : A \to \mathbb{C}$ holomorphic and z_0 a singularity of f. Then z_0 is called **removable** if f can be extended to a holomorphic function on $A \cup \{z_0\}$, *i.e.*, if there is a holomorphic function $\tilde{f} : A \cup \{z_0\} \to \mathbb{C}$ with $\tilde{f}|_A = f$.

Example 7.3.

(1) The function $f : \mathbb{C} \backslash \{1\} \to \mathbb{C}$, $z \mapsto \frac{z^2-1}{z-1}$ has a removable singularity at 1, with holomorphic extension $f(z) = z + 1$.

(2) The function $g : \mathbb{C} \backslash \{0\} \to \mathbb{C}$, $z \mapsto \frac{\sin(z)}{z}$ has a removable singularity at 0. This follows from Taylor's theorem.

Theorem 7.4 (Riemann removability condition)**.** Let $A \subseteq \mathbb{C}$ open, $f : A \to \mathbb{C}$ holomorphic, and z_0 a singularity of f. Then the following are equivalent.

(1) z_0 is removable.

(2) There exists $\delta > 0$ with $\dot{D}(z_0, \delta) \subseteq A$, such that f is bounded on $\dot{D}(z_0, \delta)$.

Proof. Assume that z_0 is removable and let $\tilde{f}$ be the holomorphic extension. There exists $\delta > 0$, such that

$$|\tilde{f}(z) - \tilde{f}(z_0)| < 1, \qquad \forall z \in D(z_0, \delta).$$

For $z \in \dot{D}(z_0, \delta)$, we have that $\tilde{f}(z) = f(z)$, and thus

$$|f(z)| \leq |f(z) - \tilde{f}(z_0)| + |\tilde{f}(z_0)| = |\tilde{f}(z) - \tilde{f}(z_0)| + |\tilde{f}(z_0)| \leq 1 + |\tilde{f}(z_0)|,$$

so that f is bounded on $\dot{D}(z_0, \delta)$. To prove the converse, suppose that f is bounded on $\dot{D}(z_0, \delta)$ for some $\delta > 0$. Define the function $g : D(z_0, \delta) \to \mathbb{C}$ by

$$g(z) = \begin{cases} (z - z_0)^2 f(z), & \text{for } z \neq z_0, \\ 0, & \text{for } z = z_0. \end{cases}$$

The function is holomorphic on $\dot{D}(z_0, \delta)$ and since f is bounded on the same set, we have that

$$\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \to z_0} (z - z_0) f(z) = 0.$$

Hence, g is differentiable in the complex sense at $z = z_0$ with $g'(z_0) = 0$. By Taylor's theorem, g has the Taylor series on $D(z_0, \delta)$ as

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!} (z - z_0)^n = \sum_{n=2}^{\infty} \frac{g^{(n)}(z_0)}{n!} (z - z_0)^n.$$

Construct the function

$$h(z) = \sum_{n=2}^{\infty} \frac{g^{(n)}(z_0)}{n!} (z - z_0)^{n-2}, \qquad \forall z \in D(z_0, \delta).$$

Clearly h is holomorphic on $D(z_0, \delta)$ and that

$$h(z) = \frac{g(z)}{(z - z_0)^2} = f(z), \qquad \forall z \in \dot{D}(z_0, \delta).$$

We can therefore set

$$\tilde{f}(z) = \begin{cases} h(z), & \text{for } z \in D(z_0, \delta), \\ f(z), & \text{for } z \notin D(z_0, \delta), \end{cases}$$

which is a holomorphic extension of f to $A \cup \{z_0\}$. □

Definition 7.5. Let $A \subseteq \mathbb{C}$ open, $f : A \to \mathbb{C}$ holomorphic, and z_0 a singularity of f. We say that z_0 is a **pole** of f, if there exists $\delta > 0$ with $D(z_0, \delta) \subseteq A \cup \{z_0\}$ and a holomorphic function $g : D(z_0, \delta) \to \mathbb{C}$ with $g(z_0) \neq 0$, such that

$$f(z) = \frac{g(z)}{(z - z_0)^m}, \qquad \forall z \in \dot{D}(z_0, \delta), \text{ for some } m \in \mathbb{N}.$$

The number m is called the **pole order** of f at z_0. If $m = 1$, z_0 is called a **simple pole** of f.

Remark 7.6. Importantly, the order of a pole is well-defined. Suppose that we have two holomorphic functions $g : D(z_0, \delta_1) \to \mathbb{C}$ and $h : D(z_0, \delta_2) \to \mathbb{C}$ with $\delta_1, \delta_2 > 0$ and $g(z_0) \neq 0 \neq h(z_0)$. If we have that

$$\frac{g(z)}{(z - z_0)^{m_1}} = f(z) = \frac{h(z)}{(z - z_0)^{m_2}}, \qquad \forall z \in \dot{D}(z_0, \min\{\delta_1, \delta_2\}),$$

then if $m_1 > m_2$, we can find $(z - z_0)^{m_1 - m_2} = g(z)/h(z)$ in a small deleted disk around z_0. Letting $z \to z_0$, we will find that $g(z_0) = 0$, leading to a contradiction. A similar contradiction can be deduced for $m_1 < m_2$.

Theorem 7.7. Let $A \subseteq \mathbb{C}$ open, $f : A \to \mathbb{C}$ holomorphic, and z_0 a singularity of f. Then the following are equivalent.

(1) z_0 is a pole of f.

(2) $|f(z)| \to \infty$ as $z \to z_0$.

Proof. Assume that z_0 is a pole of f, and let

$$f(z) = \frac{g(z)}{(z - z_0)^m}, \qquad \text{for } z \in \dot{D}(z_0, \delta),$$

with g, m, and δ as in the definition. Since $g(z_0) \neq 0$ and $m \geq 1$, the continuity of g at z_0 implies that

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \frac{|g(z)|}{|z - z_0|^m} = \infty.$$

To prove the converse, assume that $|f(z)| \to \infty$ as $z \to z_0$. There exists a deleted disk $\dot{D}(z_0, \delta) \subseteq A$ on which f has no zero. Therefore, $1/f$ is holomorphic and bounded on that disk, thus can be extended to a holomorphic function $\tilde{f}$ on $D(z_0, \delta)$, fulfilling $\tilde{f}(z_0) = 0$. By Taylor's theorem, we have that

$$\tilde{f}(z) = \sum_{\nu=m}^{\infty} a_\nu (z - z_0)^\nu, \qquad a_\nu \in \mathbb{C},$$

where $m \in \mathbb{N}$ and $a_m \in \mathbb{C} \setminus \{0\}$. We can therefore write

$$\tilde{f}(z) = (z - z_0)^m h(z),$$

where $h : D(z_0, \delta) \to \mathbb{C}$ is holomorphic and does not have zeros in $D(z_0, \delta)$, fulfilling $h(z_0) = a_m \neq 0$. We find that

$$f(z) = \frac{1}{\tilde{f}(z)} = \frac{\frac{1}{h(z)}}{(z - z_0)^m} = \frac{g(z)}{(z - z_0)^m},$$

where $g = 1/h$ is holomorphic on $D(z_0, \delta)$ fulfilling $g(z_0) \neq 0$. □

Definition 7.8. Let $A \subseteq \mathbb{C}$ open, $f : A \to \mathbb{C}$ holomorphic, and z_0 a singularity of f. If z_0 is neither a pole, nor a removable singularity, then it is called an **essential singularity**.

Theorem 7.9 (Casorati-Weierstrass)**.** Let $A \subseteq \mathbb{C}$ open and $f : A \to \mathbb{C}$ holomorphic. Let z_0 be an essential singularity of f and assume that $\dot{D}(z_0, \delta) \subseteq A$ for $\delta > 0$. Then for every $w \in \mathbb{C}$ and every $\epsilon > 0$, there exists $z \in \dot{D}(z_0, \delta)$, such that $|f(z) - w| < \epsilon$.

Proof. Assume for contradiction that there exists $w \in \mathbb{C}$ and $\epsilon > 0$, such that

$$|f(z) - w| \geq \epsilon, \qquad \forall z \in \dot{D}(z_0, \delta).$$

Therefore, the function

$$g : \dot{D}(z_0, \delta) \to \mathbb{C}, \qquad z \mapsto \frac{1}{f(z) - w}$$

is holomorphic, bounded, and has no zeros. By the Riemann removability condition, we can extend g to a holomorphic function on $D(z_0, \delta)$. There are two cases.

- **Case I:** $g(z_0) \neq 0$. Then $f(z) = \frac{1}{g(z)} + w$ holds on all of $D(z_0, \delta)$, making f holomorphic on $D(z_0, \delta)$. In other words, z_0 is a removable singularity, leading to a contradiction.
- **Case II:** $g(z_0) = 0$. Since g is not identically zero on $D(z_0, \delta)$, we can write

  $$g(z) = (z - z_0)^m h(z), \qquad m \in \mathbb{N},$$

  with some holomorphic function $h : D(z_0, \delta) \to \mathbb{C}$ with $h(z_0) \neq 0$. This follows from Taylor's theorem and the identity theorem. For $z \in \dot{D}(z_0, \delta)$, we have that

  $$f(z) = \frac{1}{g(z)} + w = \frac{1}{(z - z_0)^m h(z)} + w = \frac{G(z)}{(z - z_0)^m},$$

  where we define $G(z) = \frac{1}{h(z)} + w(z - z_0)^m$ for $z \in D(z_0, \delta)$. Now $G(z_0) = \frac{1}{h(z_0)} \neq 0$, so f has a pole at z_0, leading to a contradiction. □

Remark 7.10. (1) The converse to the Cosorati-Weierstrass theorem is true as well: If the image of every deleted disk around z_0 is dense in $\mathbb{C}$, then z_0 cannot be removable or a pole.

(2) The **Picard's theorem** refines the Cosorati-Weierstrass theorem: It asserts that under the same assumptions, for every $\delta > 0$ with $\dot{D}(z_0, \delta) \subseteq A$, there exists some $c(z_0, \delta) \in \mathbb{C}$, such that $\mathbb{C} \setminus \{c(z_0, \delta)\} \subseteq f(\dot{D}(z_0, \delta))$. This means that $f(\dot{D}(z_0, \delta))$ is not only dense, but is always all of $\mathbb{C}$ except for possibly one point.

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7.2 Laurent Decomposition

We will now see how to find a series expansion of holomorphic functions around an isolated singularity, the **Laurent expansion**. This will also give us a helpful characterization of the type of singularity under consideration. To simplify notation, we introduce for $0 \le r < R \le \infty$ the **annulus**

$$\mathscr{A}_r^R(z_0) = \{z \in \mathbb{C};\ r < |z - z_0| < R\}.$$

Note that $\mathscr{A}_0^R(z_0) = \dot{D}(z_0, R)$. We restrict our attention to the case $z_0 = 0$ and write $\mathscr{A}_r^R = \mathscr{A}_r^R(0)$.

Theorem 7.11 (Laurent decomposition)**.** Let $0 \le r < R \le \infty$ and let $f : \mathscr{A}_r^R \to \mathbb{C}$ be holomorphic. Then f has the decomposition

$$f(z) = g(z) + h(z^{-1}), \qquad \forall z \in \mathscr{A}_r^R,$$

where $g : D(0,R) \to \mathbb{C}$ is holomorphic, and $h : D(0, r^{-1}) \to \mathbb{C}$ is holomorphic with $h(0) = 0$. The decomposition is unique. The function $z \mapsto h(z^{-1})$ is the **principal part** of the Laurent decomposition.

Lemma 7.12. Let $0 \le r < \rho < P < R \le \infty$ and assume that $G : \mathscr{A}_r^R \to \mathbb{C}$ is holomorphic. Then we have that

$$\int_{\partial D(0,\rho)} G(z)dz = \int_{\partial D(0,P)} G(z)dz.$$

Proof. Consider the notation as is shown in Figure 7.

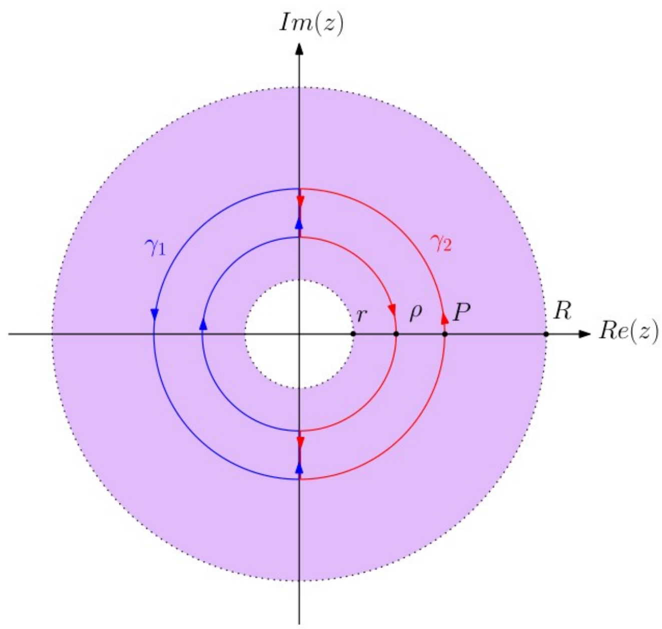

Figure 7: The curves γ_1 and γ_2 contained in the annulus $\mathscr{A}_r^R$.

We see that γ_1 and γ_2 are both closed and contained in simply connected domains $\mathscr{A}_r^R \backslash [0,\infty)$ and $\mathscr{A}_r^R \backslash (-\infty, 0]$, respectively. Thus,

$$\int_{\gamma_1} G(z)dz + \int_{\gamma_2} G(z)dz = 0,$$

which implies the result. $\square$

Proof of Theorem 7.11. We first show the uniqueness. Assume that

$$f(z) = g(z) + h(z^{-1}) = \tilde{g}(z) + \tilde{h}(z^{-1}), \qquad \forall z \in \mathscr{A}_r^R,$$

with g, h and $\tilde{g}, \tilde{h}$ as required in the theorem. We set $G = g - \tilde{g}$ and $H = h - \tilde{h}$, so $G(z) = -H(z^{-1})$ for all $z \in \mathscr{A}_r^R$. We define the function $F : \mathbb{C} \to \mathbb{C}$ by

$$F(z) = \begin{cases} G(z), & |z| < R, \\ -H(z^{-1}), & |z| > r, \end{cases}$$

which is well-defined and entire. Let $\rho \in (r, R)$, then on the compact set $\overline{D(0,\rho)}$, the continuous function $F(z) = G(z)$ is bounded. The continuous function H is also bounded on the compact set $\overline{D(0,\rho^{-1})}$, therefore $F(z) = -H(z^{-1})$ is bounded on $\overline{D(0,\rho)}$. In total, F is bounded and entire, thus must be constant by Liouville's theorem. Furthermore, we have that

$$\lim_{z\to\infty} |F(z)| = \lim_{z\to\infty} |H(z^{-1})| = 0,$$

since $h(0) = \tilde{h}(0) = 0$ and $H = h - \tilde{h}$. It follows then $F \equiv 0$ on $\mathbb{C}$, and so $G \equiv 0$ and $H \equiv 0$, indicating that g and h are unique. Now we turn to the existence of the Laurent decomposition. By the uniqueness property, it suffices to show the existence on every annulus A_ρ^P for $r < \rho < P < R$. Fix $z \in \mathscr{A}_\rho^P$ and define the function $G : \mathscr{A}_\rho^P \to \mathbb{C}$ by setting

$$G(w) = \begin{cases} \frac{f(w)-f(z)}{w-z}, & w \neq z, \\ f'(z), & w = z, \end{cases}$$

which is continuous on $\mathscr{A}_\rho^P$ and holomorphic on $\mathscr{A}_\rho^P \setminus \{z\}$. By the Riemann removability condition, we see that G is in fact holomorphic on $\mathscr{A}_\rho^P$, and we find that

$$\begin{aligned} \int_{\partial D(0,P)} \frac{f(w)}{w-z} dw - f(z) \int_{\partial D(0,P)} \frac{dw}{w-z} &= \int_{\partial D(0,P)} G(w) dw \\ &= \int_{\partial D(0,\rho)} G(w) dw = \int_{\partial D(0,\rho)} \frac{f(w)}{w-z} dw - f(z) \int_{\partial D(0,\rho)} \frac{dw}{w-z}, \end{aligned}$$

by the previous lemma. Note that the mapping $w \mapsto \frac{1}{w-z}$ is holomorphic for $w \neq z$, and since $|z| > \rho$, by Cauchy's integral theorem, we have that

$$\int_{\partial D(0,\rho)} \frac{dw}{w-z} = 0.$$

Moreover, since $|z| < P$, by Cauchy's integral formula, we have that

$$\int_{\partial D(0,P)} \frac{dw}{w-z} = 2\pi i.$$

Using these two results, we have that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \left(\int_{\partial D(0,P)} \frac{f(w)}{w-z} dw - \int_{\partial D(0,\rho)} \frac{f(w)}{w-z} dw \right) = g(z) + h(z^{-1}), \\ g(z) &= \frac{1}{2\pi i} \int_{\partial D(0,P)} \frac{f(w)}{w-z} dw, \quad |z| < P, \qquad h(z) = -\frac{1}{2\pi i} \int_{\partial D(0,\rho)} \frac{f(w)}{w-z^{-1}} dw, \quad 0 < |z| < \rho^{-1}. \end{aligned}$$

As in the proof of the generalized Cauchy integral formula, we see that g can be extended to a holomorphic function on $D(0, P)$, whereas h can be extended to a holomorphic function on $\dot{D}(0, \rho^{-1})$. We show that h can be extended to a holomorphic function by $h(0) = 0$. Indeed, for $w \in \partial D(0,\rho)$ and $|z| \in \dot{D}(0,\rho^{-1})$, we have that

$$|w - z^{-1}| \geq \big| |z^{-1}| - |w| \big| = ||z^{-1}| - \rho| > 0.$$

Using the standard bound for integrals, we thus have that

$$|h(z)| \leq \frac{1}{2\pi} \frac{\max_{w\in\partial D(0,\rho)} |f(w)|}{|z^{-1}| - \rho} \cdot 2\pi\rho \to 0, \qquad \text{as } z \to 0.$$

In particular, h is bounded for z near 0, and by the Riemann removability condition, there is a removable singularity at $z = 0$, so by continuity $h(0) = 0$. $\square$

Theorem 7.13 (Laurent expansion theorem)**.** Let $z_0 \in \mathbb{C}$, $0 \leq r < R \leq \infty$, and $f : \mathscr{A}_r^R(z_0) \to \mathbb{C}$ holomorphic. Then f has a unique Laurent expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

and both series on the right-hand side of the equation converge absolutely on $\mathscr{A}_r^R(z_0)$ and uniformly absolutely on every compact subset of $\mathscr{A}_r^R(z_0)$. Moreover, we have the representation formulas for $\rho \in (r, R)$ as

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_{\partial D(z_0,\rho)} \frac{f(w)}{(w - z_0)^{n+1}} dw, \qquad n \in \mathbb{N}, \\ b_n &= \frac{1}{2\pi i} \int_{\partial D(z_0,\rho)} f(w)(w - z_0)^{n-1} dw, \qquad n \in \mathbb{N}. \end{aligned}$$

Proof. We may assume without loss of generality that $z_0 = 0$. By the Laurent decomposition theorem, we have that $f(z) = g(z) + h(z^{-1})$ and let

$$g(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \forall z \in D(0, R), \qquad h(z) = \sum_{n=1}^{\infty} b_n z^n, \quad \forall z \in D(0, r^{-1}),$$

which are the Taylor series of g and h (note that $h(0) = 0$). The uniqueness then follows from the uniqueness of the Laurent decomposition. The claims on convergence follow trivially. Now we show the representation formula. Let $n \in \mathbb{N}$, and note that by Taylor's theorem, we have that

$$a_n = \frac{g^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_{\partial D(0,\rho)} \frac{g(w)}{w^{n+1}} dw, \qquad \rho \in (0, R).$$

Now let $\rho > r$. Note that the mapping $w \mapsto w^{-1}$ maps $\partial D(0, \rho)$ to $\partial D(0, \rho^{-1})$ and reverts the direction of the parametrization of the circle, so

$$\int_{\partial D(0,\rho)} \frac{h(w^{-1})}{w^{n+1}} dw = -\int_{\partial D(0,\rho^{-1})} \frac{h(\xi)}{\xi^{-n-1}} d(\xi^{-1}) = \int_{\partial D(0,\rho^{-1})} h(\xi)\xi^{n-1} d\xi.$$

The last integral vanishes by the Cauchy integral theorem, since $\rho^{-1} < r^{-1}$ and the map $\xi \mapsto h(\xi)\xi^{n-1}$ is holomorphic for $n \geq 0$[9]. In total, we obtain the formula for a_n, and for b_n we can argue analogously. ☐

Example 7.14. Consider the function

$$f : \mathbb{C} \setminus \{1, 3\} \to \mathbb{C}, \qquad z \mapsto \frac{2}{z^2 - 4z + 3}.$$

We want to calculate its Laurent series in the annulus $\mathscr{A}_1^3$. For this, we write

$$f(z) = \frac{1}{1 - z} + \frac{1}{z - 3}.$$

By the formula for geometric series, we have that

$$\begin{aligned} \frac{1}{1-z} &= = -\frac{1}{z} \frac{1}{1 - \frac{1}{z}} = -\frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}}, \qquad \text{for } |z| > 1, \\ \frac{1}{z-3} &= -\frac{1}{3} \frac{1}{1 - \frac{z}{3}} = -\frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n = -\sum_{n=0}^{\infty} \frac{z^n}{3^{n+1}}, \qquad \text{for } |z| < 3. \end{aligned}$$

Thus the Laurent series of f in $\mathscr{A}_1^3$ is given by

$$f(z) = -\sum_{n=0}^{\infty} \frac{z^n}{3^{n+1}} - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = g(z) + h(z^{-1}).$$

[9]For $n = 0$, use that $h(0) = 0$ and use the Riemann removability condition.

Theorem 7.15. Let $A \subseteq \mathbb{C}$ be open, $f : A \to \mathbb{C}$ be holomorphic, and z_0 be a singularity of f. Let $R > 0$ be such that $\mathscr{A}_0^R = \dot{D}(z_0, R) \subseteq A$, and consider the Laurent series of f on $\dot{D}(z_0, R)$, which exists by the Laurent decomposition theorem. In this situation, we have that

(1) z_0 is removable if and only if $b_n = 0$ for all $n \geq 1$.

(2) z_0 is a pole of order $m \geq 1$ if and only if $b_n = 0$ for all $n > m$ and $b_m \neq 0$.

(3) z_0 is essential if and only if $b_n \neq 0$ for infinitely many $n \geq 1$.

Proof. The proof is left as an exercise and will be ignored here. ☐

Definition 7.16. Let $A \subseteq \mathbb{C}$ be open, $f : A \to \mathbb{C}$ be holomorphic, and z_0 be an isolated singularity of f. Let $R > 0$ with $\dot{D}(z_0, R) \subseteq A$. The coefficient b_1 in the Laurent expansion is called the **residue** of f at z_0. We write

$$b_1 = \mathrm{Res}(f; z_0).$$

8 The Residue Theorem

Recall that if $A \subseteq \mathbb{C}$ is a simply connected domain and $f : A \to \mathbb{C}$, Cauchy's integral theorem would yield that

$$\int_\gamma f(z)dz = 0$$

for every closed, piecewise continuously differentiable curve $\gamma : [a, b] \to \mathbb{C}$, $a < b$, with $\gamma([a, b]) \subseteq A$. The **residue theorem** allows us to calculate a wider class of integrals, namely, if $f : A \backslash \{z_1, \cdots, z_k\} \to \mathbb{C}$ is holomorphic with $z_1, \cdots, z_k \in D$ pairwise distinct, we will show that for any closed piecewise continuously differentiable curve $\gamma : [a, b] \to \mathbb{C}$, $a < b$, with $\gamma([a, b]) \subseteq A\backslash \{z_1, \cdots, z_k\}$, we have that

$$\int_\gamma f(z)dz = 2\pi i \sum_{j=1}^{k} I(\gamma; z_j)\mathrm{Res}(f; z_j),$$

where $I(\gamma; z_j)$ is the **index** or **winding number** of γ with respect to z_j (which we will introduce below) and $\mathrm{Res}(f; z_j)$ is the **residue** of f at z_j.

8.1 Calculation of Residues

Throughout this section, let $A \subseteq \mathbb{C}$ open, $f : A \to \mathbb{C}$ holomorphic, and z_0 a singularity of f. We give some rules to calculate the residue.
In some cases, the Laurent expansion of a function is explicit, and we can read off the residue explicitly.

Example 8.1. (1) $\mathrm{Res}\left(\frac{\cos(z)}{z}; 0\right) = 1$, since $\frac{\cos(z)}{z} = \frac{1}{z} - \frac{z}{2!} + \cdots$ for $z \in \mathbb{C}\backslash \{0\}$.

(2) $\mathrm{Res}\left(\exp\left(\frac{1}{z}\right); 0\right) = 1$, since $\exp\left(\frac{1}{z}\right) = \sum_{n=0}^\infty \frac{1}{n! z^n}$ for $z \in \mathbb{C}\backslash \{0\}$.

(3) Analogously, we have that $\mathrm{Res}\left(\exp\left(\frac{1}{z^k}\right); 0\right) = 0$ for $k \in \mathbb{N}\backslash \{1\}$.

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Lemma 8.2. Let $R > 0$ be such that $\dot{D}(z_0, R) \subseteq A$.

(1) For $\rho \in (0, R)$, we have that

$$b_1 = \frac{1}{2\pi i} \int_{\partial D(z_0, \rho)} f(z)dz.$$

(2) If z_0 is removable, then $\mathrm{Res}(f; z_0) = 0$.

Proof. (1) is just as defined in Laurent expansion theorem. (2) directly follows from the characterization of removable singularities via the Laurent series. ☐

Lemma 8.3. Suppose we can write $f : \dot{D}(z_0, R) \to \mathbb{C}$ as

$$f(z) = \frac{g(z)}{(z - z_0)^m}, \qquad z \in \dot{D}(z_0, R),$$

where $g : D(z_0, R) \to \mathbb{C}$ is holomorphic and fulfills $g(z_0) \neq 0$, and $m \in \mathbb{N}$. Then

$$\operatorname{Res}(f; z_0) = \frac{g^{(m-1)}(z_0)}{(m-1)!}.$$

In particular, for a simple pole (where $m = 1$), we have that

$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Proof. Since g is holomorphic on $D(z_0, R)$, we can apply Taylor's theorem and obtain that

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!} (z - z_0)^n, \qquad z \in D(z_0, R),$$

so we obtain for $z \neq z_0$ the Laurent series of f as

$$f(z) = \frac{g(z)}{(z - z_0)^m} = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!} (z - z_0)^{n-m}.$$

We can read off the residue as the term corresponding to $n = m - 1$, that is,

$$\operatorname{Res}(f; z_0) = \frac{g^{(m-1)}(z_0)}{(m-1)!}.$$

For $m = 1$, we have (using the continuity of g) that

$$\operatorname{Res}(f; z_0) = g(z_0) = \lim_{z \to z_0} g(z) = \lim_{z \to z_0} (z - z_0) f(z).$$

□

Corollary 8.4. Let g and h be holomorphic functions in $D(z_0, R)$ with $g(z_0) \neq 0$, $h(z_0) = 0$, and $h'(z_0) \neq 0$. Then $f = g/h$ has a residue at z_0 given by

$$\operatorname{Res}(f; z_0) = \frac{g(z_0)}{h'(z_0)}.$$

Proof. Note that f has a simple pole in z_0. Indeed, we have that

$$\lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{h(z)}{z - z_0} = h'(z_0) \neq 0,$$

so the mapping $z \mapsto \begin{cases} \frac{z - z_0}{f(z)}, & z \neq z_0, \\ \frac{1}{h'(z_0)}, & z = z_0, \end{cases}$ is holomorphic on $D(z_0, r)$ for some $r \in (0, R)$ and we can apply the previous proposition to obtain that

$$\operatorname{Res}\left(\frac{g}{h}; z_0\right) = \lim_{z \to z_0} (z - z_0) \frac{g(z)}{h(z)} = \frac{g(z_0)}{h'(z_0)}.$$

□

Proposition 8.5. Let g and h be holomorphic in $D(z_0, R)$ with $g(z_0) \neq 0$, $h(z_0) = h'(z_0) = 0$, and $h''(z_0) \neq 0$. Then $f = g/h$ has a residue at z_0 given by

$$\operatorname{Res}(f; z_0) = \frac{2g'(z_0)}{h''(z_0)} - \frac{2g(z_0) h'''(z_0)}{3(h''(z_0))^2}.$$

Proof. The proof is left as an exercise and will be ignored here. □

There are more general formulas for obtaining the residue, which can be found in *Marsden-Hoffman, Basic Complex Analysis, 3rd Edition, Section 4.1.*

8.2 Winding Number

Definition 8.6. Let $\gamma : [a,b] \to \mathbb{C}$ be a closed, piecewise continuously differentiable curve and $z \in \mathbb{C}\backslash\gamma([a,b])$. The number

$$I(\gamma; z) = \frac{1}{2\pi i}\int_\gamma \frac{dw}{w-z}$$

is called the **index** or **winding number** of γ with respect to z. We furthermore define

$$\begin{aligned} &\mathrm{Int}(\gamma) = \{z \in \mathbb{C}\backslash\gamma([a,b]);\ I(\gamma;z) \neq 0\}\,,\ \text{the } \textbf{interior} \text{ of } \gamma, \\ \text{and }\ &\mathrm{Ext}(\gamma) = \{z \in \mathbb{C}\backslash\gamma([a,b]);\ I(\gamma;z) = 0\}\,,\ \text{the } \textbf{exterior} \text{ of } \gamma. \end{aligned}$$

With the Cauchy's integral formula, we can see that for $k \in \mathbb{Z}\backslash\{0\}$ and $z_0 \in \mathbb{C}$, consider the curve $\gamma : [0,1] \to \mathbb{C}$, $t \mapsto z_0 + r\exp(2\pi i k t)$, the winding number of γ with respect to any z with $|z - z_0| \neq r$ is

$$I(\gamma; z) = \begin{cases} 0, & |z-z_0| > r, \\ k, & |z-z_0| < r. \end{cases}$$

Therefore, at lest in the case of a circle, the winding number describes how often the curve γ travels around z in a counterclockwise fashion. A more complicated example is as shown in Figure 8.

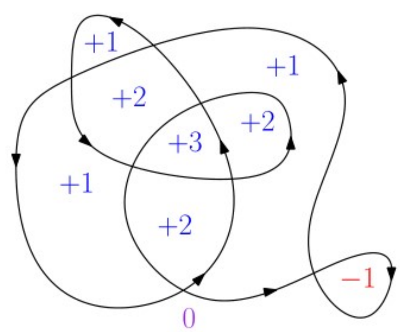

Figure 8: A curve γ with respective winding numbers

Proposition 8.7. Let $\gamma : [a,b] \to \mathbb{C}$ be a closed, piecewise continuously differentiable curve and $z_0 \in \mathbb{C}\backslash\gamma([a,b])$. Then $I(\gamma; z_0) \in \mathbb{Z}$.

Proof. We define

$$g(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z_0} ds.$$

At all points t where the integrand is continuous, we have by the fundamental theorem of calculus that

$$g'(t) = \frac{\gamma'(t)}{\gamma(t) - z_0} \qquad \Longrightarrow \qquad \frac{d}{dt}\left(\exp(-g(t))(\gamma(t) - z_0)\right) = 0.$$

Therefore, the function $t \mapsto \exp(-g(t))(\gamma(t) - z_0)$ must be piecewise constant and by continuity, is constant. This means that

$$\exp(-g(b))(\gamma(b) - z_0) = \exp(-g(a))(\gamma(a) - z_0),$$

which in turn implies that $\exp(-g(b)) = \exp(-g(a))$ since γ is closed. Note that $g(a) = 0$ by definition, so we have that $\exp(-g(b)) = 1$, which implies that $g(b) \in 2\pi i\mathbb{Z}$. The result then follows trivially. $\square$

Proposition 8.8. Let $\gamma : [a,b] \to \mathbb{C}$ be a closed, piecewise continuously differentiable curve, and assume that $z_0, z_1 \in \mathbb{C}\backslash\gamma([a,b])$ can be connected by a continuous path, *i.e.*, there exists $\phi : [0,1] \to \mathbb{C}\backslash\gamma([a,b])$ with $\gamma(0) = z_0$ and $\gamma(1) = z_1$. Then, we have that $I(\gamma; z_0) = I(\gamma; z_1)$.

Proof. The set $\mathbb{C}\backslash\gamma([a,b])$ is open. Consider the set[10]

$$C(z_0) = \{z \in \mathbb{C}\backslash\gamma([a,b]);\ z \text{ can be connected to } z_0 \text{ by a continuous path in } \mathbb{C}\backslash\gamma([a,b])\}.$$

By construction, $C(z_0)$ is path-connected, and therefore also connected. The map

$$I(\gamma;\cdot) : \mathbb{C}\backslash\gamma([a,b]) \to \mathbb{C}, \qquad z \mapsto I(\gamma; z)$$

is holomorphic, and thus also continuous. Therefore also $I(\gamma; C(z_0))$ is connected, but $I(\gamma; C(z_0)) \subseteq \mathbb{Z}$ consists of isolated points. Therefore, $I(\gamma; C(z_0))$ can only consist of a single point, and therefore $I(\gamma; z_0) = I(\gamma; z_1)$. $\square$

8.3 The Residue Theorem and Its Proof

Theorem 8.9 (The residue theorem)**.** Let $A \subseteq \mathbb{C}$ be a simply connected domain and $z_1, \cdots, z_k$ be finitely many pairwise distinct points in A. Let $f : A\backslash\{z_1, \cdots, z_k\} \to \mathbb{C}$ be holomorphic, and $\gamma : [a,b] \to \mathbb{C}$, $a < b$ with $\gamma([a,b]) \subseteq A\backslash\{z_1, \cdots, z_k\}$ be a closed, piecewise continuously differentiable curve. Then we have that

$$\int_\gamma f(z)dz = 2\pi i \sum_{j=1}^{k} I(\gamma; z_j)\mathrm{Res}(f; z_j).$$

Proof. By definition, all z_j, $j \in 1, \cdots, k$ are (isolated) singularities of f. For some $R_j > 0$, we therefore have a Laurent expansion

$$f(z) = \sum_{n=1}^{\infty} \frac{b_{n,j}}{(z - z_j)^n} + \sum_{n=0}^{\infty} a_{n,j}(z - z_j)^n, \qquad z \in \dot{D}(z_j, R_j).$$

Note that the mapping $\mathbb{C}\backslash\{z_j\} \to \mathbb{C}$, $z \mapsto \sum_{n=1}^{\infty} \frac{b_{n,j}}{(z-z_j)^n}$ is holomorphic, since the principal part in the Laurent decomposition theorem is holomorphic on $D\left(z_0, \frac{1}{r}\right)$ and in this case $r = 0$. We can therefore set

$$g : A\backslash\{z_1, \cdots, z_k\} \to \mathbb{C}, \qquad z \mapsto f(z) - \sum_{j=1}^{k}\sum_{n=1}^{\infty} \frac{b_{n,j}}{(z - z_j)^n},$$

which is holomorphic and has (isolated) singularities at $z_1, \cdots, z_k$. For every $j = 1, \cdots, k$, the Laurent expansion of g in $\dot{D}(z_j, R_j)$ has a vanishing principal part[11], so by the characterization of removable singularities via the Laurent series, we can see that all singularities are removable. By removing the singularities one by one, we obtain a holomorphic function $\tilde{g} : A \to \mathbb{C}$ with $\tilde{g}|_{A\backslash\{z_1,\cdots,z_k\}} = g$. Since A is simply connected, we can use Cauchy's integral theorem for simply connected domains and obtain that

$$0 = \int_\gamma g(z)dz = \int_\gamma \left(f(z) - \sum_{j=1}^{k}\sum_{n=1}^{\infty} \frac{b_{n,j}}{(z - z_j)^n}\right) dz = \int_\gamma f(z)dz - \sum_{j=1}^{k} \int_\gamma \left(\sum_{n=1}^{\infty} \frac{b_{n,j}}{(z - z_j)^n}\right) dz.$$

The Laurent series in the above expression converge uniformly on every compact set contained in $A\backslash\{z_1, \cdots, z_k\}$ in view of the Laurent decomposition theorem. Thus, we can further exchange the order of summation and integration. Hence, we would obtain that

$$\begin{aligned} 0 &= \int_\gamma f(z)dz - \sum_{j=1}^{k}\sum_{n=1}^{\infty} b_{n,j} \int_\gamma \frac{1}{(z - z_j)^n} dz \\ &= \int_\gamma f(z)dz - \sum_{j=1}^{k} b_{1,j} \int_\gamma \frac{1}{z - z_j} dz = \int_\gamma f(z)dz - \sum_{j=1}^{k} \mathrm{Res}(f; z_j) \cdot 2\pi i I(\gamma; z_j). \end{aligned}$$

since $z \mapsto \dfrac{1}{(z - z_j)^n}$ have primitive for $n \geq 2$

$\square$

[10]This is the **connected component** of z_0 in $\mathbb{C}\backslash\gamma([a,b])$.

[11]We use that the Laurent coefficients are additive.

8.4 Applications of the Residue Theorem

Definition 8.10. Let $A \subseteq \mathbb{C}$ be open, $f : A \to \mathbb{C}$ be holomorphic, and z_0 be a pole or removable singularity such that f is not constant to zero on some $\dot{D}(z_0, R)$, $R > 0$. Consider the Laurent expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \qquad z \in \dot{D}(z_0, R).$$

We define the **zero order**

$$0\text{-ord}(f; z_0) = \begin{cases} \min \{n \in \mathbb{N};\ a_n \neq 0\}, & \text{if there is no } m \geq 1 \text{ with } b_m \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and the **pole order**

$$\infty\text{-ord}(f; z_0) = \begin{cases} 0, & \text{if there is no } m \geq 1 \text{ with } b_m \neq 0, \\ \max \{n \in \mathbb{N};\ b_n \neq 0\}, & \text{otherwise.} \end{cases}$$

Remark 8.11. The notion of the pole order above generalizes the one given in the definition of pole. See also the characterization of pole via the Laurent series.

Theorem 8.12 (Root-pole counting theorem)**.** Let $A \subseteq \mathbb{C}$ be a simply connected domain and $f : A \setminus \{z_1, \cdots, z_k\} \to \mathbb{C}$ be holomorphic and non-zero, with pairwise distinct zeros[12] or poles $z_1, \cdots, z_k \in A$. Suppose that $\gamma : [a, b] \to \mathbb{C}$, $a < b$ is a closed, piecewise continuously differentiable curve with $\gamma([a, b]) \subseteq A \setminus \{z_1, \cdots, z_k\}$. Then we have the **root-pole counting formula**

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{j=1}^{k} I(\gamma; z_j)(0\text{-ord}(f; z_j) - \infty\text{-ord}(f; z_j)).$$

Proof. The function $z \mapsto \frac{f'(z)}{f(z)}$ is holomorphic on $A \setminus \{z_1, \cdots, z_k\}$, and using the residue theorem yields

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{j=1}^{k} I(\gamma; z_j) \text{Res}\left(\frac{f'}{f}; z_j\right).$$

Hence, it suffices to show that if z_0 is a zero or pole of f, then

$$\text{Res}\left(\frac{f'}{f}; z_0\right) = 0\text{-ord}(f; z_0) - \infty\text{-ord}(f; z_0) =: m \in \mathbb{Z}.$$

In both cases, there exists $R > 0$ and $g : D(z_0, R) \to \mathbb{C}$ holomorphic with $g(z_0) \neq 0$, such that

$$f(z) = (z - z_0)^m g(z), \qquad z \in \dot{D}(z_0, R).$$

Indeed, if z_0 is a pole this follows by definition, and if z_0 is a zero, we use Taylor's theorem for the continuation of f to obtain such a g. For $z \in \dot{D}(z_0, R)$, we find that

$$\frac{f'(z)}{f(z)} = \frac{m(z - z_0)^{m-1} g(z) + (z - z_0)^m g'(z)}{(z - z_0)^m g(z)} = \frac{m}{z - z_0} + \frac{g'(z)}{g(z)}, \qquad z \in \dot{D}(z_0, R).$$

Since g is holomorphic on $D(z_0, R)$ and $g(z_0) \neq 0$, we have that g'/g is holomorphic on the some disk $D(0, \delta)$ with $\delta < R$. It then follows that

$$\text{Res}\left(\frac{f'}{f}; z_0\right) = \lim_{z \to z_0} (z - z_0) \frac{f'(z)}{f(z)} = m.$$

$\square$

[12] This is understood as removable singularities in which the holomorphic continuation attains the value zero.

Corollary 8.13 (The argument principle)**.** Under the same hypotheses as the root-pole counting theorem, we have that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} = N(0) - N(\infty),$$

where

$$N(0) = \sum_{j=1}^k 0\text{-ord}(f; z_j), \qquad N(\infty) = \sum_{j=1}^k \infty\text{-ord}(f; z_j),$$

if every zero or pole has winding number 1 for γ.

Remark 8.14. The name "argument principle" for the previous corollary comes with the following observation: Let $\gamma : [a, b] \to \mathbb{C}$, $a < b$ be a closed, piecewise continuously differentiable curve with $\gamma([a, b]) \subseteq A\backslash\{z_1, \cdots, z_k\}$, and $f : A\backslash\{z_1, \cdots, z_k\} \to \mathbb{C}$ be such that f has no zeros on $A\backslash\{z_1, \cdots, z_k\}$, where $z_1, \cdots, z_k$ are poles or zeros. We consider $w_0 = f(\gamma(a))$ and ask how much the argument of $w_t = f(\gamma(t))$ is changed when moving continuously along the curve $[a, b] \to \mathbb{C}$, $t \mapsto f(\gamma(t))$. That is, we are interested in

$$\Delta_\gamma \arg(f) = 2\pi I(f \circ \gamma; 0).$$

Note that the latter is well-defined since f does not have zeros on $\gamma([a, b])$. Without loss of generality, assume that γ is piecewise continuously differentiable. We note that

$$i\Delta_\gamma \arg(f) = 2\pi i I(f \circ \gamma; 0) = \int_{f\circ\gamma} \frac{dz}{z} = \int_a^b \frac{f'(\gamma(t))\gamma'(t)dt}{f(\gamma(t))} = \int_\gamma \frac{f'(z)}{f(z)}dz.$$

Combined with the argument principle, we can see that if every zero or pole of f has winding number 1 of for γ, we can compute that

$$\Delta_\gamma \arg(f) = 2\pi(N(0) - N(\infty)).$$

Theorem 8.15 (Rouché's theorem)**.** Let $A \subseteq \mathbb{C}$ be a simply connected domain, $f, g : A \to \mathbb{C}$ e holomorphic, and $\gamma : [a, b] \to \mathbb{C}$ be a closed, piecewise continuously differentiable curve with $\gamma([a, b]) \subseteq A$ which surrounds every point in its interior exactly once[13]. Assume that f and g have finitely many zeros[14] in A and

$$|f(z) - g(z)| < |f(z)|, \qquad z \in \gamma([a, b]).$$

Then f and g have no zeros on $\gamma([a, b])$ and have the same number of zeros in $\text{Int}(\gamma)$, counting multiplicities.

Proof. Let $\lambda \in [0, 1]$. We consider the interpolation $h_\lambda = f + \lambda(g - f) : A \to \mathbb{C}$ between f and g. Note that for $z \in \gamma([a, b])$, we have

$$\begin{aligned} |h_\lambda(z)| &\geq |f(z)| - \lambda|g(z) - f(z)| > |f(z)|(1 - \lambda) \geq 0, \quad \lambda \neq 0, \\ |h_0(z)| &= |f(z)| > |f(z) - g(z)| \geq 0. \end{aligned}$$

So h_λ has no zeros on $\gamma([a, b])$. By the root-pole counting formula, we can see that

$$\frac{1}{2\pi i}\int_\gamma \frac{h_\lambda'(z)}{h_\lambda(z)}dz = \frac{1}{2\pi i}\int_\gamma \frac{f'(z) - \lambda(g'(z) - f'(z))}{f(z) - \lambda(g(z) - f(z))}dz = N_\lambda(0),$$

where $N_\lambda(0)$ counts the number of zeros of h_λ in $\text{Int}(\gamma)$. By considering the integral expression, we see that $\lambda \mapsto N_\lambda(0)$ must be continuous on $[0, 1]$, but since $N_\lambda(0) \in \mathbb{Z}$, it is necessarily constant and we must have $N_0(0) = N_1(0)$. ☐

Rouché's theorem is useful for locating the zeros of functions by comparison with functions that are simpler to analyze.

Example 8.16. Consider the polynomial $P(z) = z^4 - 6z + 3$. How many zeros does it have in $\mathscr{A}_1^2(0)$? Take $\gamma_{\partial D(0,2)} : [0, 2\pi] \to \mathbb{C}$, $t \mapsto 2\exp(it)$, and take $f(z) = z^4$, $g(z) = P(z) = z^4 - 6z + 3$. For $z \in \partial D(0, 2)$,

$$|f(z) - g(z)| = |6z - 3| \leq 15 < 16 = |f(z)|,$$

thus both f and g have 4 zeros inside $D(0, 2)$. Noe consider $\gamma_{\partial D(0,1)} : [0, 2\pi] \to \mathbb{C}$, $t \mapsto \exp(it)$, and define $\tilde{f}(z) = -6z$, $g(z) = P(z) = z^4 - 6z + 3$. For $z \in \partial D(0, 1)$,

$$|\tilde{f}(z) - g(z)| = |z^4 + 3| \leq 4 < 6 = |\tilde{f}(z)|,$$

thus both $\tilde{f}$ and g have 1 zero inside $D(0, 1)$. Hence we can conclude that $P = g$ has 3 zeros in the annulus $\mathscr{A}_1^2(0)$.

[13]This means that for all $z \in A\backslash\gamma([a, b])$, $I(\gamma; z) \in \{0, 1\}$.

[14]This condition is in fact not necessary.

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We give another application of Rouché's theorem, which is known as Hurwitz' theorem.

Theorem 8.17 (Hurwitz' theorem)**.** Let $A \subseteq \mathbb{C}$ be a domain and $(f_n)_{n\in\mathbb{N}}$ be a sequence of holomorphic functions $f_n : A \to \mathbb{C}$ such that $f_n \overset{\text{u}}{\to} f$ on every compact $K \subseteq A$ for some $f : A \to \mathbb{C}$. Assume that f is not identically zero, and let $z_0 \in A$. Then $f(z_0) = 0$ if and only if there is a sequence $(z_n)_{n\in\mathbb{N}} \subseteq A$ with $z_n \to z_0$ and $N \in \mathbb{N}$, such that $f_n(z_n) = 0$ for all $n \geq N$.

Proof. ($\Longrightarrow$) First we note that f is holomorphic on A by the Weierstrass approximation theorem. Assume that f is not identically equal to zero. We first show that if $\gamma : [a, b] \to \mathbb{C}$, $a < b$ with $\gamma([a, b]) \subseteq B\setminus \{z \in A;\ f(z) = 0\}$ (with $B \subseteq A$ a simply connected domain) is a closed curve avoiding all zeros of f, then there exists $N(\gamma) \in \mathbb{N}$ such that every f_n with $n \geq N(\gamma)$ has the same number of zeros in $\mathrm{Int}(\gamma)$ as f. Indeed, since $|f|$ is continuous and non-zero on the compact set $\gamma([a, b])$, we have that

$$|f(z)| \geq \min_{w\in\gamma([a,b])} |f(w)| =: m > 0, \qquad \text{for all } z \in \gamma([a, b]).$$

Clearly $f_n \overset{\text{u}}{\to} f$ on $\gamma([a, b])$, so there exists an $N(\gamma)$ with

$$|f_n(z) - f(z)| < m \leq |f(z)|, \qquad \text{for all } z \in \gamma([a, b]),\ n \geq N(\gamma).$$

We apply Rouché's theorem to conclude that f_n and f have the same number of zeros in $\mathrm{Int}(\gamma)$ for $n \geq N(\gamma)$. Now suppose that $f(z_0) = 0$. The zeros of f are isolated by some previous proposition, meaning that there is a number $\delta > 0$, such that $f(z) \neq 0$ for $z \in \dot{D}(z_0, \delta)$. For each $k \in \mathbb{N}$, consider

$$\gamma_{\partial D\left(z_0,\frac{\delta}{k}\right)} : [0, 2\pi] \to \mathbb{C}, \qquad t \mapsto z_0 + \frac{\delta}{k}\exp(it).$$

Consider $N_k = N(\gamma_{\partial D\left(z_0,\frac{\delta}{k}\right)})$, then for $n \geq N_k$, we have $|z_n - z_0| \leq \frac{\delta}{k}$, which proves the theorem by setting $N = N_1$ and choosing $z_n \in D\left(z_0, \frac{\delta}{k}\right)$ for $n \geq N_k$.
($\Longleftarrow$) Conversely, assume that $f_n(z_n) = 0$ for a sequence $(z_n)_{n\in\mathbb{N}} \subseteq A$ with $z_n \to z_0$ as $n \to \infty$. Let $\epsilon > 0$, then by uniform continuity (note that $(z_n)_{n\in\mathbb{N}} \subseteq \overline{D(z_0, \delta)}$ for some $\delta > 0$) there exists an $N \in \mathbb{N}$, such that

$$|f(z_n) - f_n(z_n)| \leq \frac{\epsilon}{2}, \qquad n \geq N.$$

By continuity, there exists $N' \in \mathbb{N}$, such that

$$|f(z_0) - f(z_n)| \leq \frac{\epsilon}{2}, \qquad n \geq N'.$$

Therefore, we can deduce that

$$|f(z_0) - f_n(z_n)| \leq |f(z_0) - f(z_n)| + |f(z_n) - f_n(z_n)| \leq \epsilon, \qquad n \geq \max\{N, N'\}.$$

This proves that $0 = \lim_{n\to\infty} f_n(z_n) = f(z_0)$. □

A special case of Hurwitz' theorem (sometimes stated under the same name) is the following: Assume that $f_n \overset{\text{u}}{\to} f$ on every compact set $K \subseteq A$ as $n \to \infty$, and f_n has no zeros on A. Then either $f \equiv 0$ or f has no zeros on A. The following corollary is an important application of Hurwitz' theorem.

Corollary 8.18. Let $A \subseteq \mathbb{C}$ a domain and $(f_n)_{n\in\mathbb{N}}$ a sequence of injective holomorphic functions $f_n : A \to \mathbb{C}$, such that $f_n \overset{\text{u}}{\to} f$ on every compact $K \subseteq A$ for some $f : A \to \mathbb{C}$. Then f is either constant or injective.

Proof. Assume that f is not constant and let $z_0 \in A$ be fixed. The functions $z \mapsto f_n(z) - f_n(z_0)$ have no zeros in $A\setminus \{z_0\}$ by injectivity. By Hurwitz' theorem, $z \mapsto f(z) - f(z_0)$ is either identically zero on $A\setminus \{z_0\}$ or it has no zeros there. The first case is excluded by assumption, so we see that $f(z) \neq f(z_0)$ on $A\setminus \{z_0\}$. Since $z_0 \in A$ was arbitrary, we obtain the injectivity of f on A. □

8.5 Evaluation of Real Integrals Using the Residue Theorem

8.5.1 Rational Functions of Sine and Cosine

An important class of functions that can be treated using residue calculus are rational functions involving sin and cos. The main tool is the observation that for $z = \exp(i\theta)$, $\theta \in [0, 2\pi]$, we can write

$$\sin(\theta) = \frac{\exp(i\theta) - \exp(-i\theta)}{2i} = \frac{z - \frac{1}{z}}{2i},$$
$$\cos(\theta) = \frac{\exp(i\theta) + \exp(-i\theta)}{2} = \frac{z + \frac{1}{z}}{2}.$$

This observation leads to the following result:

Proposition 8.19. Let $P(X,Y), Q(X,Y) \in \mathbb{C}[X,Y]$ be two polynomials in two variables and consider

$$R(x,y) = \frac{P(x,y)}{Q(x,y)},$$

the quotient of P and Q. Assume that $Q(x,y) \neq 0$ for all $(x,y) \in \mathbb{R}$ with $x^2 + y^2 = 1$. Then

$$\int_0^{2\pi} R(\cos(\theta), \sin(\theta)) d\theta = 2\pi \sum_{z_0 \in D(0,1)} \operatorname{Res}(f; z_0),$$

where f is the rational function given by

$$f(z) = \frac{1}{z} R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right).$$

Proof. For $|z| = 1$, we have that $\frac{1}{z} = \overline{z}$, so $\frac{1}{2}\left(z + \frac{1}{z}\right) = \operatorname{Re}(z) = x$ and $\frac{1}{2i}\left(z - \frac{1}{z}\right) = \operatorname{Im}(z) = y$ are real. Since $1 = |z|^2 = x^2 + y^2$, we see that for $|z| = 1$ the function $Q(x,y)$ has no zeros. We can then rewrite the integral as

$$\int_0^{2\pi} R(\cos(\theta), \sin(\theta)) d\theta = \frac{1}{i} \int_0^{2\pi} R\left(\frac{\exp(i\theta) + \exp(-i\theta)}{2}, \frac{\exp(i\theta) - \exp(-i\theta)}{2i}\right) \cdot i \exp(-i\theta) \exp(i\theta) d\theta$$
$$= \frac{1}{i} \int_{\partial D(0,1)} f(z) dz.$$

The function f has only finitely many poles in $\overline{D(0,1)}$, so we can use the residue theorem to conclude that

$$\frac{1}{i} \int_{\partial D(0,1)} f(z) dz = 2\pi \sum_{z_0 \in D(0,1)} \operatorname{Res}(f; z_0),$$

where we used that for $z_0 \in D(0,1)$, we always have $I(\gamma_{\partial D(0,1)}; z_0) = 1$. $\square$

Example 8.20. We show that for $a \in D(0,1)$, we have that

$$\int_0^{2\pi} \frac{1}{1 - 2a\cos(\theta) + a^2} d\theta = \frac{2\pi}{1 - a^2}.$$

This is trivial for $a = 0$. Otherwise, we associate the rational function

$$f(z) = \frac{1}{z\left(1 + a^2 - az - \frac{a}{z}\right)} = -\frac{1}{a} \cdot \frac{1}{(z-a)\left(z - \frac{1}{a}\right)}.$$

There is exactly one pole of f in $D(0,1)$, which is simple and we have that

$$\operatorname{Res}(f; a) = \lim_{z \to a} (z-a) f(z) = -\frac{1}{a^2 - 1},$$

so the claim follows by the previous proposition.

8.5.2 Integrals Over the Whole Real Line

We recall the definition of integrability on $\mathbb{R}$, restricting our attention to continuous functions.

Definition 8.21. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then f is called **integrable over** $\mathbb{R}$ if the limis

$$\lim_{A\to\infty} \int_0^A f(x)dx \quad \text{and} \quad \lim_{B\to\infty} \int_{-B}^0 f(x)dx$$

both exist. We then define

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{A\to\infty} \int_0^A f(x)dx + \lim_{B\to\infty} \int_{-B}^0 f(x)dx.$$

If $|f|$ is integrable over $\mathbb{R}$, we say that f is **absolutely integrable over** $\mathbb{R}$. It is easy to see that absolute integrability implies integrability.

Proposition 8.22. Let $P(X) = a_n X^n + \cdots + a_0 \in \mathbb{R}[X]$ and $Q(X) = b_m X^m + \cdots + b_0 \in \mathbb{R}[X]$ be two polynomials with real coefficients and $a_n, b_m \neq 0$. Assume that $m \geq n+2$ and $Q(x) \neq 0$ for all $x \in \mathbb{R}$. We define the quotient function on $\mathbb{C}$ (except for a finite number of singularities) as

$$R(z) = \frac{P(z)}{Q(z)}.$$

Then $R(X)$ is absolutely integrable over $\mathbb{R}$, and we have that

$$\int_{-\infty}^{\infty} R(x)dx = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(R; z_j),$$

where $z_1, \cdots, z_k$ are the poles of R in the upper half plane $\mathscr{H} = \{z \in \mathbb{C}; \operatorname{Im}(z) > 0\}$.

Proof. We see that for $z \in \mathbb{C}\setminus\{0\}$, we have that

$$R(x) = \frac{P(z)}{Q(z)} = z^{n-m} \cdot \frac{\frac{a_0}{z^n} + \cdots + a_n}{\frac{b_0}{z^m} + \cdots + b_m}.$$

For $|z| \to \infty$, the second factor tends to $\frac{a_b}{b_m}$, and is in particular bounded. In other words, there are $M > 0$ and $c > 0$, such that

$$|R(z)| \leq M|z|^{n-m} \leq \frac{M}{|z|^2}, \qquad \text{for all } |z| > c.$$

The existence of the limit $\lim_{A\to\infty} \int_0^A |R(x)|dx$ follows easily, by considering separately the (compact) interval $[0, c]$ on which the continuous function R is bounded, and $[c, A]$ for $A \to \infty$ where we can use the bound that we have just shown for $|z| > c$. The existence of the limit $\lim_{B\to\infty} \int_{-B}^0 |R(x)|dx$ follows similarly. We now look for the value of the integral over the whole real line. Let $r > \max\{|z_1|, \cdots, |z_k|\}$ and consider the closed, piecewise continuously differentiable curve $\gamma_r : [-r, r+\pi]$ given by

$$\gamma_r(t) = \begin{cases} t, & t \in [-r, r], \\ r\exp(i(t-r)), & t \in (r, r+\pi]. \end{cases}$$

By our requirement on r, γ_r avoids the singularities of R. We also write as β_r the parametrization of the half-circle $\beta_r = \gamma_r|_{[r,r+\pi]}$. With the residue theorem, we can see that

$$\int_{\beta_r} R(z)dz + \int_{-r}^{r} R(x)dx = \int_{\gamma_r} R(z) = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(R; z_j),$$

since we have $I(\gamma; z_j) = 1$ for all $j = 1, \cdots, k$. Furthermore, we have that

$$\left|\int_{\beta_r} R(z)dz\right| = \left|\int_0^{\pi} R(r\exp(it)) \cdot ri\exp(it)dt\right| \leq r\int_0^{\pi} |R(r\exp(it))|dt \leq r \cdot \frac{M}{r^2} \cdot \pi \to 0, \qquad \text{as } r \to \infty,$$

which shows the claim. $\square$

Example 8.23. We want to calculate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2},$$

using the residue theorem. Note that $z^2+1=(z+i)(z-i)$. Thus, $f: z \mapsto \frac{1}{z^2+1}$ has exactly one pole in $\mathscr{H}$, namely i, and we can compute that

$$\operatorname{Res}(f;i) = \lim_{z\to i} \frac{1}{z+i} = \frac{1}{2i}$$

Using the previous proposition, we can obtain that

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i \cdot \frac{1}{2i} = \pi.$$

The previous approach can be extended beyond rational functions.

Proposition 8.24. (1) Suppose f is holomorphic on an open set containing $\overline{\mathscr{H}}$ with $\mathscr{H} = \{z \in \mathbb{C}; \operatorname{Im}(z) > 0\}$, except for a finite number of isolated singularities, whcih are not on the real axis. Furthermore, suppose that for $M, c > 0$ and $\alpha > 1$, we have that

$$|f(z)| \le \frac{M}{|z|^\alpha}, \qquad |z| > c,$$

then $f(x)$ is absolutely integrable over $\mathbb{R}$, and we have that

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f; z_j),$$

where $z_1, \cdots, z_k$ are the singularities of f in $\mathscr{H}$.

(2) If the conditions of (1) hold with $\mathscr{H}$ replaced by the lower half-plane $\mathscr{L} = \{z \in \mathbb{C}; \operatorname{Im}(z) < 0\}$, then

$$\int_{-\infty}^{\infty} f(x)dz = -2\pi i \sum_{j=1}^{k} \operatorname{Res}(f; z_j),$$

where $z_1, \cdots, z_k$ are the singularities of f in $\mathscr{L}$.

8.5.3 Integrals Involving Branch Cuts

We have so far studied integrals over functions that are holomorphic on $\mathbb{C}$ except for a finite number of isolated singularities. Functions involving non-integer powers involve logarithms, and their treatment using the residue theorem requires more sophisticated contours. We explain here the use of **keyhole-shaped contours**.

Proposition 8.25. Let $P(X), Q(X) \in \mathbb{R}[X]$ be two polynomials with real coefficients and consider the function $z \mapsto \frac{P(z)}{Q(z)}$. Assume that the denominator Q has no roots on the positive real axis $\mathbb{R}^+$, $R(0) \neq 0$, and $\lim_{x\to\infty} x^\lambda |R(x)| = 0$ for $\lambda \in \mathbb{R}\backslash\mathbb{Z}$, $\lambda > 0$. Then the function $x \mapsto x^{\lambda-1}R(x)$ is absolutely integrable on $\mathbb{R}^+$, and we have that

$$\int_0^\infty x^{\lambda-1}R(x)dx = \frac{\pi}{\sin(\pi\lambda)} \sum_{z_0 \in \mathbb{C}_+} \operatorname{Res}(f; z_0),$$

where $f(z) = (-z)^{\lambda-1}R(z) = \exp((\lambda-1)\operatorname{Log}(-z))R(z)$, and $\mathbb{C}_+ = \mathbb{C}\backslash[0,\infty)$.

Proof. We remark that the function f is holomorphic on $\mathbb{C}_+$ except for a finite number of poles since Log is holomorphic on $\mathbb{C}_-$. For $r > 0$, we consider the contour $\gamma_r = \gamma_{1,r} * \gamma_{2,r} * \gamma_{3,r} * \gamma_{4,r}$, where the curves $\gamma_{j,r}$, $1 \le j \le 4$ are given up to translation of the parameter intervals by

$$\begin{cases} \gamma_{1,r}(t) = \exp(i\phi)t, & t \in [r^{-1}, r], \\ \gamma_{2,r}(t) = r\exp(it), & t \in [\phi, 2\pi - \phi], \\ \gamma_{3,r}(t) = -\exp(-i\phi)t, & t \in [-r, -r^{-1}], \\ \gamma_{4,r}(t) = r^{-1}\exp(i(2\pi - t)), & t \in [\phi, 2\pi - \phi], \end{cases}$$

where $\phi \in (0, \pi)$ and $r > 1$, as is shown in Figure 9.

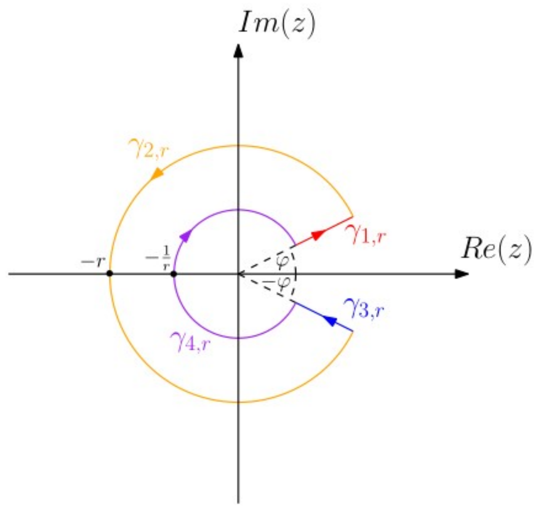

Figure 9: The curves $\gamma_{1,r}$, $\gamma_{2,r}$, $\gamma_{3,r}$, and $\gamma_{4,r}$, constituting the keyhole shape

Since $\mathbb{C}_+$ is a simply connected domain, for $r > 1$ large enough we can use the residue theorem to obtain (for ϕ small enough) that

$$\int_{\gamma_r} f(z)dz = \sum_{j=1}^{4} \int_{\gamma_{j,r}} f(z)dz = 2\pi i \sum_{z_0 \in \mathbb{C}_+} \text{Res}(f; z_0).$$

For this fixed r, we take the limit $\phi \to 0$ and by definition of $(-z)^{\lambda-1}$, the integrals over $\gamma_{1,r}$ and $\gamma_{3,r}$ converge to

$$\exp(-(\lambda-1)\pi i) \int_{1/r}^{r} x^{\lambda-1} R(x)dx \quad \text{and} \quad -\exp((\lambda-1)\pi i) \int_{1/r}^{r} x^{\lambda-1} R(x)dx,$$

respectively. The other two integrals can be shown to vanish as $r \to \infty$. □

Example 8.26. For $\lambda \in (0, 1)$, we have that

$$\int_0^\infty \frac{x^{\lambda-1}}{1+x} dx = \frac{\pi}{\sin(\pi\lambda)}.$$

9 Conformal mappings

9.1 Motivation

Let $A \subseteq \mathbb{C}$ be open and consider $f : A \to \mathbb{C}$ holomorphic. We are interested in the local mapping behavior of f. Intuitively we have for $z_0 \in A$ that

$$f(z) - f(z_0) = f'(z_0)(z - z_0) + O((z - z_0)^2) = |f'(z_0)| \cdot \exp\left(i\text{Arg}(f'(z_0))(z - z_0) + O((z - z_0)^2)\right),$$

so the map f involves a local rotation around the point z_0 of size $\text{Arg}(f'(z_0))$ and a stretching with a factor $|f'(z_0)|$.

Definition 9.1. Let $A \subseteq \mathbb{C}$ be open and consider a map $f : A \to \mathbb{C}$.

(1) We say that f **preserves angles at** $z_0 \in A$ if there exists $\theta \in [0, 2\pi)$ and $r > 0$, such that for every differentiable curve $\gamma : [-\epsilon, \epsilon] \to \mathbb{C}$ with $\epsilon > 0$, $\gamma([-\epsilon, \epsilon]) \subseteq A$, and

$$\gamma(0) = z_0, \qquad \gamma'(0) \neq 0,$$

the curve $\delta : [-\epsilon, \epsilon] \to \mathbb{C}$ given by $\delta(t) = f(\gamma(t))$ is differentiable in $t = 0$ and setting $u = \delta'(0)$ and $v = \gamma'(0)$, we have that

$$|u| = r|v|, \qquad \arg(u) = \arg(v) + \theta.$$

(2) f is **conformal at** $z_0 \in A$ if it is holomorphic and fulfills $f'(z_0) \neq 0$. f is **conformal** if it is conformal at every $z_0 \in A$.

Lemma 9.2. If $f : A \to \mathbb{C}$ is conformal at $z_0 \in A$, it preserves angles at z_0.

Proof. From the chain rule, we have that

$$u = \delta'(0) = f'(z_0)\gamma'(0) = f'(z_0)v,$$

so $\arg(u) = \arg(f'(z_0)) + \arg(v)$ and $|u| = |f'(z_0)||v|$. $\square$

Definition 9.3. Let $A, B \subseteq \mathbb{C}$ be open. A map $f : A \to B$ is called **biholomorphic**[15] if

(1) f is bijective.

(2) f is holomorphic.

(3) f^{-1} is holomorphic.

Two domains $A_1, A_2 \subseteq \mathbb{C}$ are **conformally equivalent** if there is a biholomorphic map $f : A_1 \to A_2$. Indeed, it is straightforward to show that conformal equivalence is an equivalence relation on the set of domains contained in $\mathbb{C}$.

Example 9.4. (1) The upper half-plane $\mathscr{H}$ and $D(0,1)$ are conformally equivalent, using the biholomorphic map

$$f : \mathscr{H} \to D(0,1), \qquad z \mapsto \frac{z-i}{z+i}.$$

Indeed, one can easily see that $\left|\frac{z-i}{z+i}\right| < 1$ if and only if $\mathrm{Im}(z) < 0$.

(2) $\mathbb{C}$ and $D(0,1)$ are not conformally equivalent, since any holomorphic function $f : \mathbb{C} \to D(0,1)$ is bounded, and thus would have to be constant by Liouville's theorem.

(3) Many further important examples for conformal maps can be found in *Marsden-Hoffman, Basic Complex Anaylsis, 3rd Edition, Section 5.2* and in particular pp. 340–341.

Lemma 9.5. Let $A, B \to \mathbb{C}$ be domains and $f : A \to B$ be a map. The following are equivalent:

(1) f is biholomorphic.

(2) f is bijective and conformal.

(3) f is bijective and holomorphic.

Proof. We show that (1) implies (2). Assume that f is biholomorphic. Then $f \circ f^{-1} = \mathbb{1}$. By the chain rule,

$$f'(f^{-1}(w))(f^{-1})'(w) = 1, \qquad w \in B.$$

In particular, $f'(f^{-1}(w)) \neq 0$ for all $w \in B$, and since f^{-1} is bijective, we also have that $f'(z) \neq 0$ for all $z \in A$, so f is conformal. Clearly, (2) implies (3), so it suffices to show that (3) implies (1). Assume that f is holomorphic and bijective. We show that f^{-1} is continuous first. Indeed, let $U \subseteq A$ be open. Since f is injective, it is not constant on any connected component of U, so we apply the open mapping theorem to see that $(f^{-1})^{-1}(U) = f(U)$ is open. Note again by injectivity that f' is not identiacally equal to zero on a subdomain of A, so the zero set $N(f')$ of f' only has isolated points and is closed in A (has no accumulation point in A) by the isolation-of-zeros property. Since f is an open mapping, also $\tilde{N} = f(N(f'))$ has only isolated points and is closed in B. Now let $w_0 \in B \backslash \tilde{N}$ with inverse image $z_0 = f^{-1}(w_0)$. Then

$$f(z) = f(z_0) + (z - z_0)\phi(z),$$

with a function $\psi : A \to \mathbb{C}$ continuous at z_0 and $\psi(z_0) = f'(z_0) \neq 0$. Set $z = f^{-1}(w)$ for $w \in B$, then

$$w = w_0 + (f^{-1}(w) - f^{-1}(w_0))\psi(f^{-1}(w)).$$

[15] In some texts, "conformal" is used synonymously with "biholomorphic".

The function $q = \psi \circ f^{-1}$ is continuous at w_0 and fulfills $q(w_0) = \psi(z_0) \neq 0$, so there exists $r > 0$, such that

$$f^{-1}(w) = f^{-1}(w_0) + \frac{w - w_0}{q(w)}, \qquad \text{for all } w \in D(w_0, r) \cap B.$$

We see that f^{-1} is differentiable at w_0 with the derivative

$$(f^{-1})'(w_0) = \frac{1}{q(w_0)} = \frac{1}{\psi(z_0)} = \frac{1}{f'(z_0)} = \frac{1}{f'(f^{-1}(w_0))}.$$

We have therefore shown that f^{-1} is continuous on B and holomorphic on $B \backslash \tilde{N}$, so by the Riemann removability condition, f^{-1} is holomorphic on B. □

9.2 The Riemann Mapping Theorem

Definition 9.6. A domain $\varnothing \neq A \subseteq \mathbb{C}$ is an **elementary domain**, if every holomorphic function $f : A \to \mathbb{C}$ has a primitive on A.

Proposition 9.7. Let $A, B \subseteq \mathbb{C}$ be domains and $f : A \to B$ be biholomorphic. If A is an elementary domain , then so is B.

Proof. Let $g : B \to \mathbb{C}$ be holomorphic. We need to show that g has a primitive $G : B \to \mathbb{C}$. To this end, consider the holomorphic map $g \circ f : A \to \mathbb{C}$. Since A is an elementary domain and since the function $(g \circ f) \cdot f' : A \to \mathbb{C}$ is holomorphic as well, and we obtain a primitive $F : A \to \mathbb{C}$ with $F' = (g \circ f) \cdot f'$. We set $G = F \circ f^{-1}$ (which is holomorphic since f is biholomorphic). Then, we have that

$$\begin{aligned} G'(w) &= (F \circ f^{-1})'(w) = F'(f^{-1}(w)) \cdot (f^{-1})'(w) = (g \circ f)(f^{-1}(w)) \cdot f'(f^{-1}(w)) \cdot (f^{-1})'(w) \\ &= g(w) \cdot (f \circ f^{-1})'(w) = g(w), \end{aligned}$$

so G is a primitive of g and thus B is an elementary domain by definition. □

Theorem 9.8 (Riemann mapping theorem)**.** Let $A \subseteq \mathbb{C}$ be an elementary domain with $A \neq \mathbb{C}$. Then A is conformally equivalent to the unit disk $D(0, 1)$.

Before we prove the Riemann mapping theorem, we shortly discuss its interpretation.

Corollary 9.9. (1) Let $\varnothing \neq A \subseteq \mathbb{C}$ be a domain. Then A is an elementary domain if and only if it is simply connected.

(2) Let $A \subseteq \mathbb{C}$ be a simply connected domain with $\varnothing \neq A \neq \mathbb{C}$. Then A is conformally equivalent to the unit disk $D(0, 1)$.

Proof. (1) If A is a simply connected domain, every holomorphic function $f : A \to \mathbb{C}$ has a primitive by Cauchy's integral theorem. On the other hand, let us assume that A is an elementary domain. By the Riemann mapping theorem, either $A = \mathbb{C}$ or A is conformally equivalent to $D(0, 1)$. Both are convex and therefore simply connected. Therefore, A is also simply connected, since homotopies are stable under biholomorphic maps.

(2) This follows from (1) and the Riemann mapping theorem. □