SIMULATION OF A TOWER OF GYROSCOPES

MODELING AND SIMULATION IN SCIENCE, ENGINEERING, AND ECONOMICS

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ABSTRACT

The gyroscope is a device that always tends to maintain its orientation stably. This paper examines whether a tower of gyroscopes (by stacking them up) still maintain a stable rotation, by means of simulation with MATLAB and mathematical analysis of the motion. Codes and examples of simulation results are available at Yao Xiao's GITHUB repository. Simulation videos and experiment results are available here.

Keywords Gyroscope · Structural Dynamics · Simulation

1 Introduction

A gyroscope is a mechanical device that can rotate at a high angular velocity around an axis that passes through a fixed point [1]. Moreover, it resists any external force, meaning that it always persists in maintaining its orientation. Suppose the angular velocity of the rims is ω, and the rate of precession, *i.e.*, the motion of the axle describing a cone, is ω_0. The combined rotation of ω and ω_0 results in the upper and lower halves of the gyroscope to experience accelerations of opposite directions. By Newton's second law of motion, the forces acting on the mass points in the upper and lower halves should also be in opposite directions. A torque M is thus required to sustain these forces, which is created by the gravitational force on the gyroscope, and such a gyroscopic motion is stable [2].

Then a natural question may arise: will such a motion remain stable if we stack up several gyroscopes, where the bottom of the axle of the upper gyroscope and the top of the axle of the lower one are bound together? A natural guess is that if they rotate in the same direction, they will stand, while if they rotate in opposite directions, the gyrotower will fall. For an even number of gyroscopes with the same angular velocity rotating reversely, the angular momentum of the system should be 0. Consider a wheel with n mass points on its rim, it follows that

$$\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega} = \sum_{k=1}^{n} mr^2 \left(\frac{\mathbf{r}_k \times \mathbf{v}_k}{r^2} \right) = \sum_{k=1}^{n} m \left(\mathbf{r}_k \times \mathbf{v}_k \right) = \sum_{k=1}^{n} \mathbf{r}_k \times m\mathbf{v}_k = \sum_{k=1}^{n} \mathbf{r}_k \times \mathbf{p}_k. \tag{1}$$

By the right-hand rule, it immediately follows from (1) that $\mathbf{L} = \mathbf{0}$ if the gyroscopes are connected straight up and rotate in opposite directions.

Here, we can see that by the law of conservation of the angular momentum, it seems that the system should not have gyroscopic property, which means that it will collapse. However, in our simulation, it turns out that they can still stand stably even for a tower of gyroscopes. To analyze this problem, we approach both by mathematical analysis and by simulating the gyroscopic motions with MATLAB. In our simulation, we consider all the links as simple networks of linear springs with dampers. Then we simulate the forces, velocities, and positions of each mass point at the ends of the links correspondingly. More details will be discussed later in this paper.

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2 Equations

We construct the gyroscope as a network of springs. There is an axle in the middle and a rim around it. Each mass point on the rim is connected to both ends of the axle by a link, as is shown in the left-hand side image of Figure 1. Each link is simulated as the system of a spring and a damper, as is shown in the right-hand side image of Figure 1.

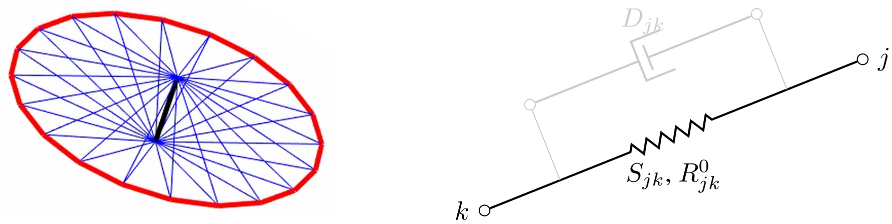

Figure 1: The construction of a gyroscope (left) and the scheme of each link consisting of a spring and a damper (right)

Now for each mass point k, let M_k denote its mass, $\mathbf{U}_k$ denote its velocity, and $\mathbf{X}_k$ denote its position. Moreover, for each link connecting mass points j and k, let S_{jk} denote the stiffness of the spring, R^0_{jk} denote the rest length of the spring, D_{jk} denote the damping constant of the damper, and T_{jk} denote the magnitude of its tension. Note that we did not use the damper in the simulation, so the following holds for this system:

$$\begin{cases} M_k \cdot \dfrac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} = \displaystyle\sum_{j \in N(k)} T_{jk} \cdot \dfrac{\mathbf{X}_j - \mathbf{X}_k}{\|\mathbf{X}_j - \mathbf{X}_k\|}, & (2) \\ \dfrac{\mathrm{d}\mathbf{X}_k}{\mathrm{d}t} = \mathbf{U}_k, & (3) \\ T_{jk} = S_{jk} \cdot (\|\mathbf{X}_j - \mathbf{X}_k\| - R^0_{jk}), & (4) \end{cases}$$

where $N(k)$ denotes the set of all neighbors of the mass point k.

3 Numerical Method

Following the previous notations, we introduce the following modified Euler method to simulate the gyrotower system modeled by (2–4):

$$\begin{cases} M_k \cdot \dfrac{\mathbf{U}_k(t+\Delta t) - \mathbf{U}_k(t)}{\Delta t} = \displaystyle\sum_{j \in N(k)} T_{jk}(t) \cdot \dfrac{\mathbf{X}_j(t) - \mathbf{X}_k(t)}{\|\mathbf{X}_j(t) - \mathbf{X}_k(t)\|}, & (5) \\ \dfrac{\mathbf{X}_k(t+\Delta t) - \mathbf{X}_k(t)}{\Delta t} = \mathbf{U}_k(t+\Delta t), & (6) \\ T_{jk} = S_{jk} \cdot (\|\mathbf{X}_j(t) - \mathbf{X}_k(t)\| - R^0_{jk}), & (7) \end{cases}$$

where $N(k)$ denotes the set of all neighbors of the mass point k. For the MATLAB implementation of this numerical method, see Section A in the Appendix.

4 Validation

To check that the simulation is correct, we compute the energy of the gyrotower system at each time step, and check that it is conserved up to approximation. The total energy of the system consist of the total kinetic energy of the mass points, the total gravitational potential energy of the mass points, and the total elastic potential energy of the links, which can be computed as

$$E = \frac{1}{2}\sum_j M_j \|\mathbf{U}_j\|^2 + \sum_j M_j g(\mathbf{X}_j)_3 + \frac{1}{2}\sum_{jk} S_{jk}(\|\mathbf{X}_j - \mathbf{X}_k\| - R^0_{jk})^2, \tag{8}$$

where the first two sums are taken over all mass points, the last sum is taken over all links, and $(\cdot)_3$ in the second sum denotes the z-coordinate of the specified vector. We simulate for 10 seconds, applying a constant horizontal force

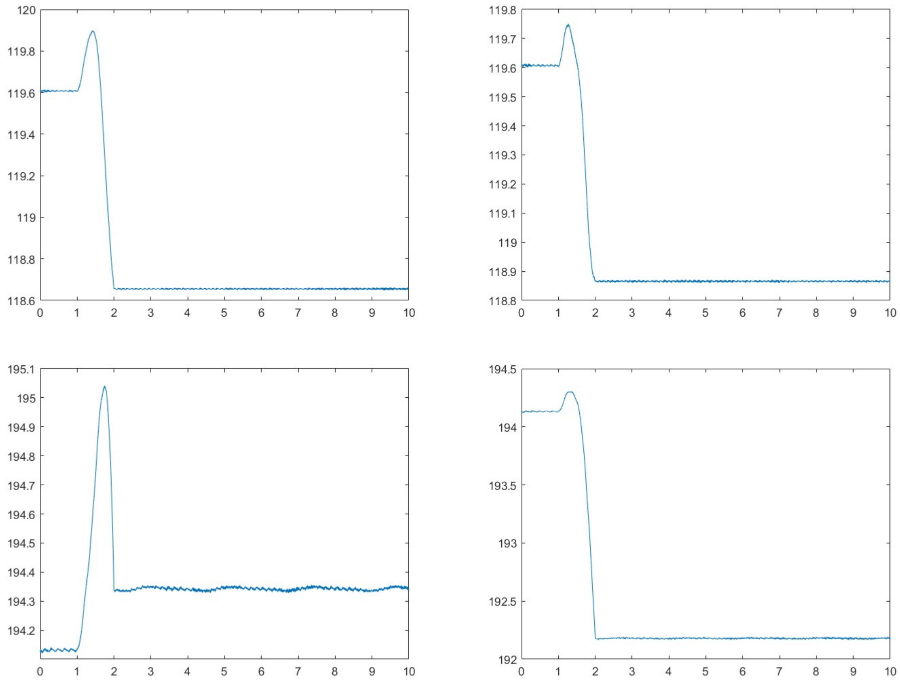

Figure 2: Total energy (J) of gyrotowers consisting of different number of gyroscopes and different rotations. The first row shows the results for two gyroscopes with the same rotating direction (left) and reverse rotating directions (right). The second row shows that for three gyroscopes.

to the top of the uppermost gyroscope from 1 to 2 seconds, and take each time step to be 2×10^{-4} seconds. In the experiments, each gyroscope has radius of 1 m and axle length of 0.5 m, with initial angular velocity of 10 rad/s. The results are shown in Figure 2.

We can see clearly that the total energy of the system is conserved up to approximation, regardless of the number of gyroscopes in the gyrotower and their rotations. Hence, we validate our simulation by the law of conservation of energy. However, note that the more gyroscopes in the gyrotower, the more accurate time stepping is required. Since simulations with too many time steps are time-consuming, we do not experiment for gyrotowers with more than three gyroscopes.

5 Results and Discussion

We want to investigate the stability of different gyrotowers against external forces. In our experiments, such stability is measured by the stability of the top of the gyrotower, *i.e.*, the z-coordinate of the top of the upmost gyroscope in the gyrotower. We will perform two experiments. The first experiment will be performed on gyrotowers consisting of two or three gyroscopes[2], either rotating in the same direction or opposite directions. Two parameters of interest are the radius of the gyroscopes and their initial angular velocity. The second experiment will be performed on gyrotowers consisting of two to seven gyroscopes, each with the same set of parameters, in order to see the effect of stacking up gyroscopes on stability. Also, the stability will be tested as we move the external force downwards. Note that all test cases use gyroscopes with axle length 0.5 m and 20 mass points on the rim.

5.1 The Effect of Radius and Angular Velocity on Stability

We experiment on gyrotowers consisting of two or three gyroscopes, since more gyroscopes would require too long simulation time. Two parameters of interest are the radius of the gyroscopes and their initial angular velocity. The experiment results are shown as in Figure 3–4. Corresponding MATLAB figures can be found here.

[2]This experiment is not performed on gyrotowers consisting of more than three gyroscopes, since the simulation would be too time-consuming (more than one hour for each simulation).

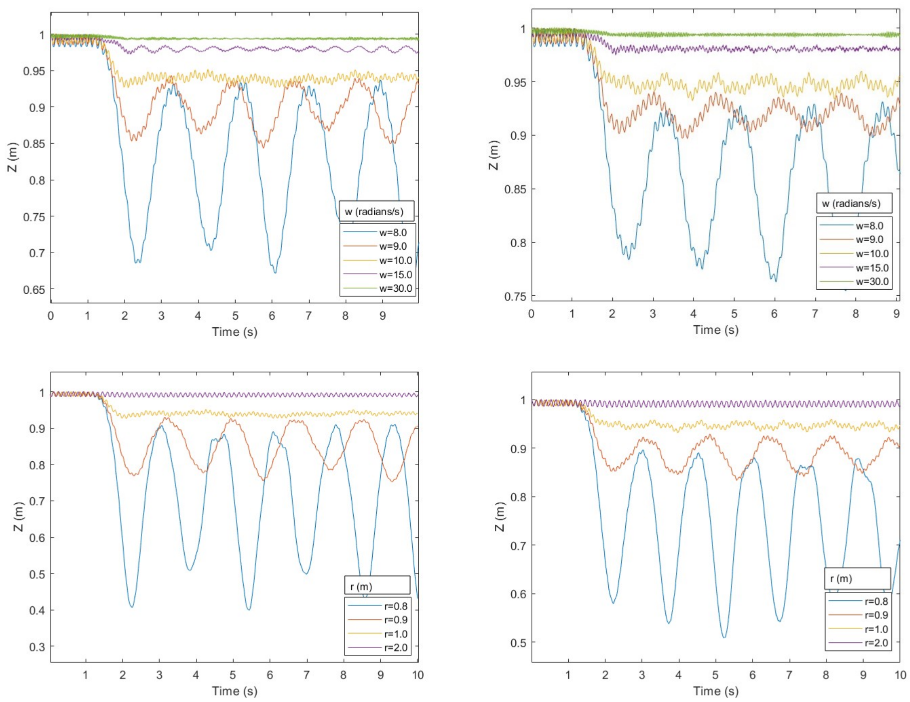

Figure 3: The stability of gyrotowers of two gyroscopes, either rotating in the same direction (left) or in opposite directions (right). The first row is tested with $r = 1.0$ m and $\omega \in [8.0, 30.0]$ radians/s. The second row is tested with $\omega = 10.0$ radians/s and $r \in [0.8, 2.0]$ m.

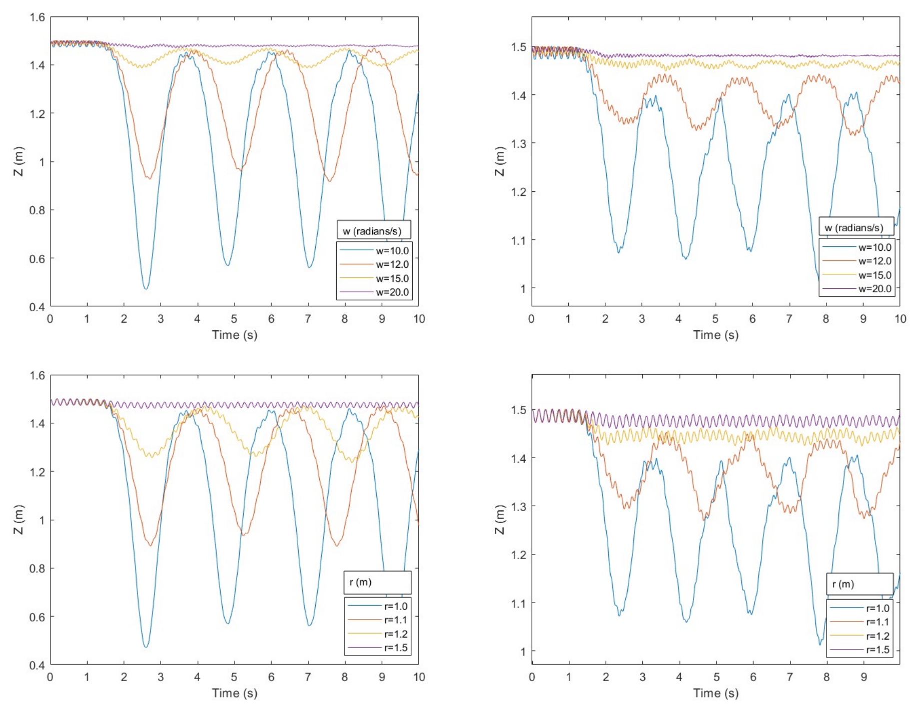

Figure 4: The stability of gyrotowers of three gyroscopes, either rotating in the same direction (left) or in opposite directions (right). The first row is tested with $r = 1.0$ m and $\omega \in [10.0, 20.0]$ radians/s. The second row is tested with $\omega = 10.0$ radians/s and $r \in [1.0, 1.5]$ m.

The plots represent the periodic motion of the gyrotowers. From the first rows of Figure 3–4, we can see that larger initial angular velocities lead to better stability of the gyrotower. The second rows also indicate that increasing the radius of the gyroscopes makes the gyrotower more stable. By comparing the left and right columns, we may further observe that gyrotowers with gyroscopes rotating all in the same direction are less stable than those with adjacent gyroscopes rotating in opposite directions. Moreover, the change of radius and angular velocity can have larger effects on the former.

Now comparing between Figure 3 and Figure 4, we further observe that the change of radius and angular velocity affects higher gyrotowers less. Indeed, in the case of two gyroscopes, the stability still increases significantly as the angular velocity ω increases from 15.0 radians/s to 30.0 radians/s, or as the radius of gyroscopes r increases from 1.0 m to 2.0 m. However, in the case of three gyroscopes, the stability hardly changes anymore for angular velocity $\omega > 20.0$ radians/s and radius of gyroscopes $r > 1.5$ m by observation[3].

In fact, we can see that gyrotowers modeled by spring networks will never fall down given radius and angular velocity sufficiently large. This is counterintuitive since torques should have canceled out at least for gyrotowers in which adjacent gyroscopes rotate in opposite directions. However, on the other hand, this is reasonable since spring axles can lean and bend, creating new torques to support the rotation. A final remark is that even though the top point of the upmost gyroscope is stable, this does not indicate that the whole gyrotower is stable. Each gyroscope on their own can lean a lot but resulting in the top to remain in a small region. The explicit behavior of each gyroscope in the gyrotower can be found in the simulation videos here.

5.2 The Effect of Stacking Up Gyroscopes on Stability

In this experiment, we test the stability of gyrotowers consisting of two to seven gyroscopes with the same set of parameters, *i.e.*, with initial angular velocity $\omega = 30.0$ radians/s and radius of gyroscopes 1.0 m. We also move the external force down from the top of the gyrotower and observe the change of stability for gyrotowers consisting of two or three gyroscopes[4]. In order to perform meaningful comparisons among gyrotowers consisting of different number of gyroscopes, we modify the z-coordinate to be divided by the original height of the gyrotower (*i.e.*, the sum of the axle lengths of all gyroscopes in the gyrotower). Furthermore, we take its standard deviation among all time steps, in order to represent stability by a single data point for clarity. The experiment results are shown as in Figure 5.

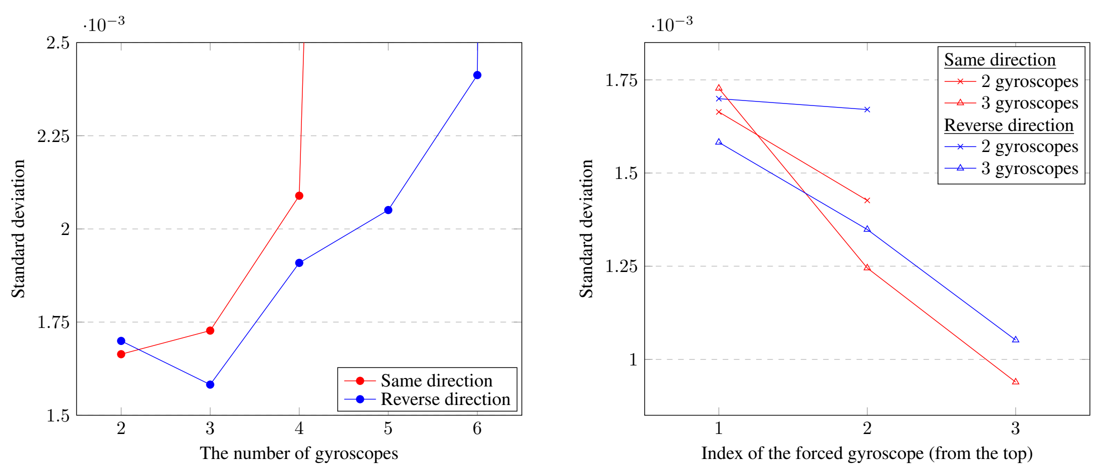

Figure 5: The change in stability as the number of gyroscopes is increased (left), and the change in stability as the external force is moved downwards.

From the left-hand side graph in Figure 5, we can see that in general, stacking up gyroscopes reduces the stability of the gyrotower. Moreover, gyrotowers with adjacent gyroscopes rotating in reverse directions resist external forces better than those with all gyroscopes rotating in the same direction. However, we also notice that the increase of number of gyroscopes may not be the only influential factor. Intuitively, as gyroscopes stack up, the instability should accumulate

[3]This observation is indeed obtained from experiments, though not plotted for clarity of the graphs. Original plots for each parameter combination can be found here under `param/`.

[4]We did not move the external force downwards for gyrotowers consisting of more than three gyroscopes, since too many parallel data points will make the graph messy. However, intuitively the behavior would be the same as gyrotowers consisting of fewer gyroscopes.

faster and faster, reflecting on the graph as the slope getting steeper and steeper. However, focusing on the blue line (reverse direction) in the left-hand side graph in Figure 5, we observe that the slope actually gets gentler when increasing the number of gyroscopes from 4 to 5. The slope even reversed when increasing the number of gyroscopes from 2 to 3. Hence, a natural guess is in addition to the number of gyroscopes, whether the number is odd or even may also be an influential factor to the stability of the gyrotower.

Now looking at the right-hand side graph in Figure 5, we move the external force down the gyrotower. The value n on the horizontal axis means that a horizontal external force is applied on the top of the axle of the nth gyroscope in the gyrotower, counting from top to bottom. As expected, with the same magnitude of forces applied, gyrotowers resist those applied at lower points better, no matter the number of gyroscopes they contain.

5.3 The Actual Angular Momentum

In all the previous discussions, we led to the conclusion that gyrotowers can stand given radius and/or angular velocity large enough. However, this is counterintuitive at least in the case of two reversely rotating gyroscopes, which we thought will collapse due to zero angular momentum of the system. Therefore, we computed the total angular momentum of the system of two reversely rotating gyroscopes, as is shown in Figure 6.

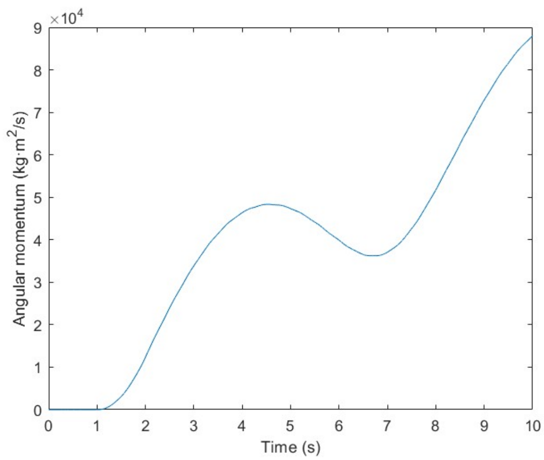

Figure 6: The magnitude of the total angular momentum of the gyrotower consisting of two reversely rotating gyroscopes

As a matter fact, the total angular momentum is **not** zero in our simulation. It remains zero only before the external force is applied at 1 s. The possible reason is that, the middle axle of the gyrotower in our simulation is not necessarily aligned. Modeled by the system of springs, the axle can bend a lot creating extra torques, which is the reason why it can resist the external force.

6 Summary and Conclusions

In this paper, we simulated and analyzed towers of gyroscopes with spring networks and structural dynamics. We validated our simulation by testing for the law of conservation of energy. Further parameter analysis is done on the radius of the gyroscopes and their initial angular velocity, and the stability of the gyrotower is positively related to both paramters. Finally, we noticed that lower gyrotowers can resist external forces better, and that arbitrary gyrotowers can resist external forces applied at lower points better. This leads to the result that gyrotowers of arbitrary height can always stand given radius and/or angular velocity sufficiently large. Finally, we discussed why out result contradicted the intuition that gyrotowers may collapse by computing the total angular momentum of the system and observing that it is not constantly zero.

References

[1] V. M. N. Passaro, A. Cuccovillo, L. Vaiani, M. De Carlo, and C. E. Campanella. 2017. Gyroscope Technology and Applications: A Review in the Industrial Perspective. *Sensors* 17, 10 (Oct. 2017), 2284. `https://doi.org/10.3390/s17102284`

[2] J. B. Scarborough. 1957. *The Gyroscope: Theory and Applications*. Interscience Publishers.

Appendix

A MATLAB Code

Here we present parts of our code essential for our simulation. Denote then number of mass points and the number of links as m and n, respectively. Some of the important variables are declared as follows.

```
int             ngyro;              % Number of gyroscopes in the gyrotower
int[n][3]       X;                  % Coordinates of points
int[m][3]       jj, kk;             % Indices of endpoints of the links
double[m][1]    S;                  % Stiffness of links
double[m][1]    Rzero;              % Rest length of links
double[n][1]    M;                  % Mass of points
int             forced;             % Index of the point where external force is applied
double[1][3]    F_ext;              % External force (kg.m/s^2)
```

Within each iteration, the numerical method for simulating the rotation of gyroscopes (5–7) is implemented as follows, with additional gravitational force and an external force applied to the system.

```
DX = X(jj, :) - X(kk, :);                       % Link vectors
DU = U(jj, :) - U(kk, :);                       % Link velocity difference vectors
R = sqrt(sum(DX .^ 2, 2));                      % Link lengths
DR = R - Rzero;                                 % Elastic changes of lengths
T = S .* DR;                                    % Link tensions
TR = T ./ R;                                    % Link tensions divided by link lengths
FF = [TR, TR, TR] .* DX;                        % Link force vectors

F = zeros(kmax, 3);                             % Initialize force array for mass points
F(:, 3) = -M * g;                               % Apply gravitational force to each link point

for link = 1 : lmax
    F(kk(link), :) = F(kk(link), :) + FF(link, :);  % Add force of the link to one end
    F(jj(link), :) = F(jj(link), :) - FF(link, :);  % Add force of the link to the other end
end

if (t > t_ext_start) && (t < t_ext_stop)        % In the specified time interval
    F(forced, :) = F(forced, :) + F_ext;        % Apply the external force
end

U = U + dt * F ./ [M, M, M];                    % Update velocities of all points
U(n + 1, :) = 0;                                % The fixed point must have velocity zero
X = X + dt * U;                                 % Update positions of all points
```

The computation of the total energy of the gyrotower system (8) is implemented as follows.

```
E = 1 / 2 * sum(M .* sqrt(sum(U .^ 2, 2)) .^ 2) ...     % Total kinetic energy
    + sum(M .* X(:, 3) * g) ...                         % Total gravitational potential energy
    + 1 / 2 * sum(S .* DR .^ 2) + EE;                   % Total elastic potential energy
```