NYU SHANGHAI
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Honors Mathematics

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Analyzing the Critical Behavior of Bernoulli Percolation in $\mathbb{Z}^3$ by Simulating the Invasion Percolation Process

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Abstract

The invasion percolation process is widely known related to the standard Bernoulli bond percolation model. It is known that the infinite cluster density $\mathbf{P}_\infty(p_c) = 0$ at critical point for Bernoulli bond percolation in $\mathbb{Z}^2$, but this conclusion is not yet proven to extend to higher dimensions. We note that it is easy to show the equivalence between $\mathbf{P}_\infty(p_c) = 0$ in $\mathbb{Z}^d$ and $G_{\mathbb{Z}^d}(0, x) \to 0$ as $|x| \to \infty$ for the invasion percolation process, where $G_{\mathbb{Z}^d}(0, x)$ is the probability that x is invaded by an invasion percolation process starting from the origin. In this paper, we will then show by simulation the dominance of $G_{\mathbb{Z}^3}(0, x)$ by $G_{\mathbb{Z}^2}(0, x)$ for the same $|x|$, based on which we show that $\mathbf{P}_\infty(p_c) = 0$ in $\mathbb{Z}^3$. Finally we will numerically estimate the fractal dimension of invasion percolation cluster in $\mathbb{Z}^3$.

Keywords

Bernoulli percolation; Invasion percolation; Infinite cluster density; Fractal dimension.

Contents

1. Introduction

Bernoulli percolation. A *percolation model* is defined by a distribution on percolation configurations on a graph $G = (V, E)$, where a *percolation configuration* $\omega = \{\omega_e;\ e \in E\}$ is an element of $\{0,1\}^E$, with $\omega_e = 1$ indicating that the edge e is *open* and $\omega_e = 0$ indicating that the edge e is *closed*. *Bernoulli bond percolation* is one of the simplest percolation model, where each edge is open with probability p and closed with probability $1-p$, each being determined independently. In the rest of the article we will use the term *Bernoulli percolation* to refer to Bernoulli bond percolation. In this paper, we will treat Bernoulli percolation on $\mathbb{Z}^d$, the d-dimensional integer lattice. Define the size of the cluster $C(x)$ of $x \in \mathbb{Z}^d$ by the number of open edges in the cluster containing x, denoted as $|C(x)|$. Then the cluster size distribution is given by

$$\mathbf{P}_n = \mathbb{P}(|C(0)| = n), \qquad n \in \mathbb{N}, \tag{1}$$

and the infinite cluster density would be

$$\mathbf{P}_\infty = \lim_{n\to\infty} \mathbf{P}_n = \lim_{n\to\infty} \mathbb{P}(|C(0)| = n). \tag{2}$$

There are many kinds of definitions of the critical point of p, all shown to be equivalent [1, 2, 3]. In this paper, we will use the version

$$p_c = \sup\{p \in [0,1];\ \mathbf{P}_\infty(p) = 0\}. \tag{3}$$

Invasion percolation. *Invasion percolation* starting at the origin on $\mathbb{Z}^d$, on the other hand, is given by the following process. Assign weights to the edges independently and identically distributed. Start with an empty set $C_0 = \emptyset$. At the first step, C_1 is obtained by adding to C_0 the edge of the smallest weight that is incident to the origin. At the second step, C_2 is obtained by adding to C_1 the edge of the smallest weight that is incident to an endpoint of the edge in C_1. Iteratively at the n-th step, C_n is obtained by adding to C_{n-1} the edge of the smallest weight that is incident to some endpoint of some edge in C_{n-1}, and so on. Let $G_{\mathbb{Z}^d}(0,x)$ denote the probability that $x \in \mathbb{Z}^d$ is invaded by an invasion percolation process starting at the origin.

Organization. The main goal of this paper is to investigate the critical behavior of Bernoulli percolation in $\mathbb{Z}^3$, and in particular discuss the possiblity of showing $\mathbf{P}_\infty(p_c) = 0$ in $\mathbb{Z}^3$. This is

done in several sections.

- In Section 2, we will present a proof that in $\mathbb{Z}^d$,

$$\mathbf{P}_\infty(p_c) = 0 \iff G_{\mathbb{Z}^d}(0,x) \to 0 \text{ as } |x| \to \infty. \tag{4}$$

  In particular, what we would like to use of this general result are that in $\mathbb{Z}^2$,

$$\mathbf{P}_\infty(p_c) = 0 \implies G_{\mathbb{Z}^2}(0,x) \to 0 \text{ as } |x| \to \infty, \tag{5}$$

  and that in $\mathbb{Z}^3$,

$$G_{\mathbb{Z}^3}(0,x) \to 0 \text{ as } |x| \to \infty \implies \mathbf{P}_\infty(p_c) = 0, \tag{6}$$

- In Section 3, we will investigate by simulation the distribution of $G_{\mathbb{Z}^2}(0,x)$ and $G_{\mathbb{Z}^3}(0,x)$ and their relation for the same $|x|$. We observe that $G_{\mathbb{Z}^3}(0,x) \leq G_{\mathbb{Z}^2}(0,x)$ for each $|x|$ (in particular for large $|x|$), so that by (5) and the fact that $\mathbf{P}_\infty(p_c) = 0$ in $\mathbb{Z}^2$ [4], we have $G_{\mathbb{Z}^2}(0,x) \to 0$ as $|x| \to \infty$, and hence $G_{\mathbb{Z}^3}(0,x) \to 0$ as $|x| \to \infty$, and by (6) we reach $\mathbf{P}_\infty(p_c) = 0$ in $\mathbb{Z}^3$, *i.e.*, the probability that there exists an infinite cluster in a critical Bernoulli percolation in $\mathbb{Z}^3$ is zero, which has not been rigorously proven yet (Section 3.1). We will further look into $G(0,x)$ in slabs which are useful in proofs (Section 3.2). Finally, we will revise a rigorous proof for supercritical Bernoulli percolation in high dimensions using slabs, and briefly mention why such proofs cannot be extended to the critical case, thus the need for the simulation (Section 3.3).

Finally in Section 4, we will numerically estimate the fractal dimension of invasion percolation cluster in $\mathbb{Z}^3$.

2. Relation Between Bernoulli and Invasion Percolations

To start with, we introduce another version of definition of invasion percolation called the *percolation cluster method*, due to the fact that the previously introduced definition is too static to provide necessary information about the dynamic growth of the invasion percolation model for proof. Let $p \in (0,1)$ be a parameter, and choose a percolation configuration ω with density p, as defined for the Bernoulli percolation. Instead of assigning all weights at the very beginning, the

percolation cluster method assigns weights to the edges only when they join the invaded region, *i.e.*, be incident to some endpoint of some edge in the invaded set of edges. If the chosen edge to invade is open, *i.e.*, $\omega_e = 1$, then the weight of the edge is assigned uniformly in $[0, p]$, and otherwise assigned uniformly in $[p, 1]$. This ensures that we always invade the edge with the smallest weight that is incident to the invaded region, thus being equivalent to the original definition.

Let w_n be the weight of the n-th edge that is invaded. For $p \in [0, 1]$, we can then define the binary random variable W_n by

$$W_n(p) := \begin{cases} 1, & \text{if } w_n \leq p, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

The empirical distribution function is then given by

$$\mathbf{W}_n(y) := \frac{1}{n} \sum_{k=1}^{n} W_k(y). \tag{8}$$

Theorem 2.1. Let p_c be the critical point of Bernoulli percolation in $\mathbb{Z}^d$. Then with probability one, we have

$$\mathbf{W}_n(p_c) \to 1 \text{ as } n \to \infty. \tag{9}$$

Remark 2.2. We will provide a sketch of the proof of this theorem in the following, building on top of the proof in [5]. With this theorem, we can see that at critical point, the invasion percolation process will invade edges with weights at most p_c with probability one as the number of invaded edges goes to infinity. This, in other words, means that at large numbers the process is unlikely to invade edges with weights exceeding p_c. Based on such an observation we can then argue the relation between the infinite cluster density of Bernoulli percolation and the invasion percolation process.

Sketch proof of Theorem 2.1. Let $\overline{p}_c$ be the critical point of the half-space percolation, and let

$$R_m := \{\, x \in \mathbb{Z}^d;\ |\, x_i| \leq m,\ \forall i = 1, \ldots, d \,\}, \tag{10}$$

the hypercube centered at the origin with side length $2m$. We consider a percolation configuration ω of density $p > \overline{p}_c$ and the time t when the corresponding invasion percolation process first

invades an edge that is incident to some vertex on the boundary of R_m. Let

$$\mathbf{M}_n(p) := \sum_{k=1}^{n}(1 - W_k(p)), \tag{11}$$

which represents the number of invaded edges that have weight larger than p, *i.e.*, that are closed, as defined in the percolation cluster method, until the n-th invasion step. In the following, we will approach the proof in a few steps.

Step 1. Show that $\mathbb{P}(\mathbf{M}_n(p) \to \infty$ as $n \to \infty) = 0$ where $p > \overline{p}_c$. In other words, this statement means that it is very unlikely that we invade closed edges at large numbers. To see this, we start by observing that for any k not exceeding the number of edges in R_m,

$$\mathbb{P}(\mathbf{M}_n(p) \to \infty \text{ as } n \to \infty \mid \mathbf{M}_t(p) \geq k) \leq 1 - \overline{\mathbf{P}}_\infty(p), \tag{12}$$

Indeed, if we denote by v the vertex on the boundary of R_m that is incident to the edge invaded at time t, the probability that v invades to infinity along a path consisting only of open edges outside of R_m is at least $\overline{\mathbf{P}}_\infty(p)$. Recall also that $\mathbf{M}_n(p)$ is the number of closed but invaded edges, thus the upper bound by $1 - \overline{\mathbf{P}}_\infty(p)$. Moreover, it is independent from which vertex v is, because the latter depends purely on the weights of edges within R_m given that v is the first time the invasion percolation touches the boundary of R_m, thus we can apply the conditioning above. As a consequence, we can see that

$$\mathbb{P}(\mathbf{M}_n(p) \to \infty \text{ as } n \to \infty) \leq (1 - \overline{\mathbf{P}}_\infty(p))\mathbb{P}(\mathbf{M}_t(p) \geq k), \tag{13}$$

and bringing $k \to \infty$ concludes that

$$\mathbb{P}(\mathbf{M}_n(p) \to \infty \text{ as } n \to \infty) = 0, \tag{14}$$

under our assumption that $p > \overline{p}_c$.

Step 2. Show that $\mathbb{P}(\mathbf{W}_n(\overline{p}_c) \to 1$ as $n \to \infty) = 1$. Since this is related to the critical behavior, we need to extend the conclusion in the first step from supercritical to critical density $p = \overline{p}_c$. To do this, we first introduce some more notations. Let K_n be the number of new edges that must be checked after invading the n-th edge, *i.e.*, the number of edges that are incident to some

endpoint of the n-th invaded edge and have not been checked before. Let

$$S_n := \sum_{k=0}^{n} K_k, \tag{15}$$

the total number of edges that have been checked until after invading the n-th edge. We now define z_n corresponding to w_n, but the weights of the n-th checked edge if we assume some deterministic order of checking edges after each invasion step. Again, we can define the binary random variable Z_n by

$$Z_n(p) := \begin{cases} 1, & \text{if } z_n \leq p, \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

The empirical distribution function is then given by

$$\mathbf{Z}_n(p) := \frac{1}{S_n} \sum_{k=1}^{S_n} Z_k(p). \tag{17}$$

Now observe that the number of edges invaded in some weight interval is always smaller than the number of edges checked in that same interval. Indeed, if the former is larger, then we must have invaded some edge without checking whether it is of the smallest possible weight, which is impossible by definition of the percolation cluster method. As a consequence, for any $p_1, p_2 \in [0, 1]$, we can see that

$$\limsup_{n\to\infty} |\mathbf{W}_n(p_1) - \mathbf{W}_n(p_2)| \leq \limsup_{n\to\infty} \frac{S_n}{n} |\mathbf{Z}_n(p_1) - \mathbf{Z}_n(p_2)|. \tag{18}$$

It is trivial that $\mathbf{Z}_n(p) \to p$ as $n \to \infty$ by the law of large numbers (see also [5]). Moreover, for each n we have $K_n \leq 2d - 1$ because one of the endpoints of an invaded edge must be the endpoint of another previously invaded edge (thus all edges incident to it must have already been checked in earlier invasion steps), and the other endpoint can have at most $2d$ incident edges in $\mathbb{Z}^d$ for check, while we have to exclude the currently invaded edge itself. Hence, we can deduce that

$$\limsup_{n\to\infty} |\mathbf{W}_n(p_1) - \mathbf{W}_n(p_2)| \leq \frac{\sum_{k=1}^{n}(2d-1)}{n} |p_1 - p_2| = (2d-1)|p_1 - p_2|. \tag{19}$$

Note that

$$\mathbf{W}_n(p) = \frac{1}{n}\sum_{k=1}^{n} W_k(p) = 1 - \frac{1}{n}\sum_{k=1}^{n}(1 - W_k(p)) = 1 - \frac{\mathbf{M}_n(y)}{n}. \tag{20}$$

Now, (14) as in the first step trivially implies that $\mathbb{P}(\mathbf{W}_n(p) \to 1 \text{ as } n \to \infty) = 1$ for any $p > \overline{p}_c$. Combined with the result in (19), we can then extend the conclusion to the critical point, such that $\mathbf{W}_n(\overline{p}_c) \to 1$ as $n \to \infty$, with probability one.

Step 3. Finally, it has been shown in the literature that $\overline{p}_c = p_c$ for any dimension. Hence, we can conclude that, with probability one, $\mathbf{W}_n(p_c) \to 1$ as $n \to \infty$ in $\mathbb{Z}^d$, as desired. □

Theorem 2.3. Let p_c be the critical point of Bernoulli percolation in $\mathbb{Z}^d$. Then

$$\mathbf{P}_\infty(p_c) = 0 \iff G_{\mathbb{Z}^d}(0, x) \to 0 \text{ as } |x| \to \infty, \tag{21}$$

where $G_{\mathbb{Z}^d}(0, x)$ is the probability that x is invaded by an invasion percolation process starting from the origin.

Remark 2.4. What in particular is useful about this theorem is that, we know that $\mathbf{P}_\infty(p_c) = 0$ in $\mathbb{Z}^2$ [4], so this theorem would imply that $G_{\mathbb{Z}^2}(0, x) \to 0$ as $|x| \to \infty$. Then by comparing $G_{\mathbb{Z}^d}(0, x)$, $d \geq 2$ with $G_{\mathbb{Z}^2}(0, x)$ for the same $|x|$, we can possibly show that $G_{\mathbb{Z}^d}(0, x) \to 0$ as $|x| \to \infty$ for certain higher dimensions. Then by the other direction of the theorem, we can conclude that $\mathbf{P}_\infty(p_c) = 0$ in those dimensions.

Sketch proof of Theorem 2.3. We will show the forward direction directly and the backward direction by its contrapositive statement.

Forward. $\mathbf{P}_\infty(p_c) = 0 \implies G_{\mathbb{Z}^d}(0, x) \to 0$ **as** $|x| \to \infty$. Assume that $\mathbf{P}_\infty(p_c) = 0$, *i.e.*, with zero probability will there be an infinite cluster at the origin in Bernoulli percolation at critical point. Take a sequence of $x \in \mathbb{Z}^d$ such that $|x| \to \infty$, and assume for contradiction that there exists a path from the origin to each x with all edges on the path having weight at most p_c. If we consider the percolation cluster method, these edges must all be open, so that as $|x| \to \infty$ we have a path of all open edges connecting the origin to x, forming an infinite cluster at the origin, which is contradictory to the assumption that $\mathbf{P}_\infty(p_c) = 0$. Then as $|x| \to \infty$, any path in $\mathbb{Z}^d$ connecting the origin to x must have at least one edge with weight exceeding p_c. By Theorem 2.1,

we know that the invasion percolation process will invade edges with weights at most p_c with probability one as the number of invaded edges goes to infinity. In other words, it is asympotically unlikely that invade any path starting from the origin towards x as $|x| \to \infty$. Hence, we can conclude that $G_{\mathbb{Z}^d}(0, x) \to 0$ as $|x| \to \infty$, as desired.

Backward. $\mathbf{P}_\infty(p_c) = 0 \Longleftarrow G_{\mathbb{Z}^d}(0, x) \to 0$ as $|x| \to \infty$. We prove the contrapositive statement. Assume that $\mathbf{P}_\infty(p_c) > 0$, then there is positive probability that the origin is in an infinite cluster. By definition of $\mathbf{P}_\infty(p_c)$, translation invariance is trivial so that we any point in $\mathbb{Z}^d$ has positive probability of being in an infinite cluster. In particular, let $x \in \mathbb{Z}^d$ be an arbitrary point in the infinite cluster, then $G_{\mathbb{Z}^d}(0, x) > 0$, *i.e.*, x will be invaded by the invasion percolation process starting at the origin with positive probability. Indeed, there exists a path of open edges that connects the origin to x, and if we use the percolation cluster method, all edges on this path would have weight at most p_c due to openness. By Theorem 2.1, we know that the invasion percolation process will invade edges with weights at most p_c with probability one as the number of invaded edges goes to infinity. Hence, we can conclude x will always be finally invaded with positive probability, *i.e.*, $G_{\mathbb{Z}^d}(0, x) > 0$. Pick a sequence of such x with $|x| \to \infty$ (this is possible because we are in an infinite cluster), then we have shown that

$$\mathbf{P}_\infty(p_c) > 0 \implies G_{\mathbb{Z}^d}(0, x) \not\to 0 \text{ as } |x| \to \infty, \tag{22}$$

so taking its contrapositive statement gives the desired result. □

3. Simulation of Invasion Percolations

In this section, we will simulate the invasion percolation process on $\mathbb{Z}^2$ and $\mathbb{Z}^3$ at critical point p_c, intending to compare $G_{\mathbb{Z}^2}(0, x)$ and $G_{\mathbb{Z}^3}(0, x)$ for the same $|x|$ (Section 3.1). Then, we will simulate the invasion percolation process in slabs $\mathbb{Z}^2 \times \{ x_3 \in \mathbb{Z};\ |x_3| \leq l \}$ and compare the results with that in $\mathbb{Z}^2$ and $\mathbb{Z}^3$ (Section 3.2). This is useful because the invasion percolation process is usually investigated in slabs and then extended to the whole of $\mathbb{Z}^d$ space. Finally, we will provide a brief sketch proof of the exponential decay of the cluster size distribution $\mathbf{P}_n$ of a supercritical Bernoulli percolation (*i.e.*, $p > p_c$) in $\mathbb{Z}^d$, using the invasion percolation analysis presented in [6] (Section 3.3). The proof, however, cannot be intuitively extended to the critical behavior of Bernoulli percolation in $\mathbb{Z}^d$ ($d \geq 3$), which is the reason why we use the simulation

Algorithm 1 Simulation of invasion percolation on $\mathbb{Z}^d$.

```
x_0 ← (0, ..., 0);  // Initial vertex
S_0 ← { (0, ..., 0) };  // Set of invaded vertices
C_0 ← ∅;  // Set of invaded edges
C'_0 ← ∅;  // Set of candidate edges for the next invasion step
while i = 1, 2, ... do
    C'_i ← C'_{i-1};
    for e ∈ {edges with x_{i-1} as one of its endpoints} \ C_{i-1} \ C'_{i-1} do
        Pick a weight uniformly at random in [0, 1] and assign it to e;
        C'_i ← C'_i ∪ { e };
    end for
    Pick e_i ∈ C'_i with the smallest weight w_i;
    C_i ← C_{i-1} ∪ { e_i };
    C'_i ← C'_i \ { e_i };
    x_i ← the other endpoint of e_i;
end while
```

approach to investigate the critical Bernoulli percolation in $\mathbb{Z}^3$.

3.1. Invasion Percolation on $\mathbb{Z}^2$ and $\mathbb{Z}^3$

Algorithm. It is clear that it is impossible to algorithmically assign weights to all the edges in the infinite space $\mathbb{Z}^d$, thus the previous definitions of the invasion percolation process, including the percolation cluster method, are not directly applicable. However, thanks to the independence of the weights of the edges, it does not matter in which order or at which time we assign the weights to the edges, as long as they are ready when included into the boundary of the invasion. In particular, let C_n be the set edges that we have included during the invasion percolation process until the n-th step. The boundary ∂C_n of C_n is the set of edges that are incident to some endpoint of some edge in C_n but not in C_n. At the $(n+1)$-th step, all the weights that we need to check before invading the next edge are the weights of the edges in $\overline{C_n} = C_n \cup \partial C_n$, so it suffices to make sure that all the weights of the edges in $\overline{C_n}$ are assigned before the $(n+1)$-th invasion step, and that the weights of previously assigned edges do not change in further steps. One can easily see that this process is equivalent to the previous definitions by independence. The algorithm is shown as in Algorithm 1.

Results in two dimensions. We simluate the invasion percolation process on $\mathbb{Z}^2$ and as shown in Figure 1 are the visualization results using two different random seeds. The simulation is done for 10^6 steps, but we visualize only 10^4 steps since otherwise the visualization of the graph would be too dense and cluttered. The point marked in red is the origin $0 \in \mathbb{Z}^2$ where the

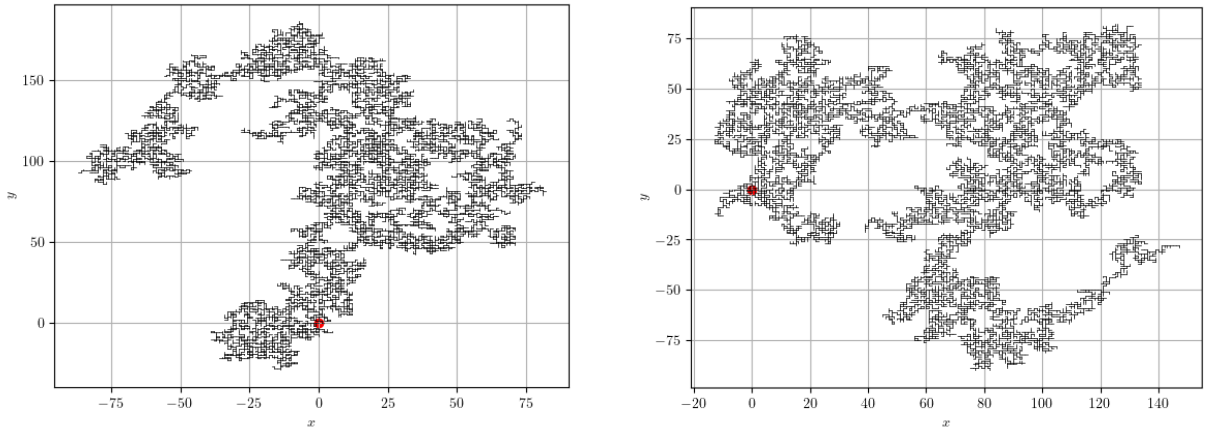

Figure 1: Simulation of invasion percolation in $\mathbb{Z}^2$ for 10^4 steps.

invasion percolation process begins, and the black lines are the edges that have been included into the invaded region, *i.e.*, C_{10^4}. As we can see, the invasion percolation process differs a lot with different random seeds, going in different major directions and forming different fractal-like shapes.

One particular aspect of interest is the distribution of $G_{\mathbb{Z}^2}(0, x)$. Existing work has exploited that $G(0, x)$ follows a power law with respect to $|x|$ in $\mathbb{Z}^2$, as well as similar results in $\mathbb{Z}^3$ and $\mathbb{Z}^4$ [7, 8]. However, these results are based on assumptions of finite-sized boxes, special behaviors when hitting the box boundaries, and more. Here we focus on the infinite space of $\mathbb{Z}^2$ without any constraints, except that we simulate only 10^6 steps and obtain 300 samples for computing the probabilities due to technical constraints.

As is shown in Figure 2 is the distribution of $G_{\mathbb{Z}^2}(0, x)$ with $0 < |x| \leq 100$. The reason for limiting $|x| \leq 100$ is that larger $|x|$ may suffer from marginal effects due to the constraints in the number of steps of simulation, *i.e.*, that 10^6 steps of simulation may not have reached a steady state for larger distances. As we can see from the fitted curve (where the fitting is done using the non-linear least squares method), the distribution of $G_{\mathbb{Z}^2}(0, x)$ respects the power law $G_{\mathbb{Z}^2}(0, x) \sim |x|^{-\alpha}$, with $\alpha \approx 0.13$ by simulation. This is consistent with the conclusions of [7, 8], though with a smaller estimated α value. This is, however, reasonable because we do not impose stopping conditions until hitting the maximum number of simulation steps, thus having much larger opportunities to explore the space around the origin, leading to higher probabilities of invasion and thus slower decay of $G_{\mathbb{Z}^2}(0, x)$.

Results in three dimensions. We also simluate the invasion percolation process on $\mathbb{Z}^3$ and as shown in Figure 3 with the same setup as in $\mathbb{Z}^2$. The simulation is done for 10^6 steps, but we

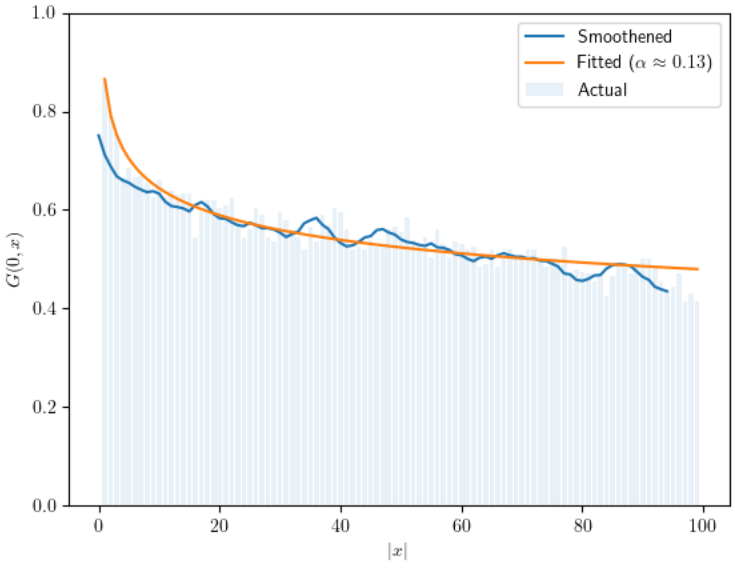

Figure 2: Distribution of $G_{\mathbb{Z}^2}(0,x)$ with 10^6 steps of invasion percolation in $\mathbb{Z}^2$, with each probability $G_{\mathbb{Z}^2}(0,x)$ computed from 300 independent samples.

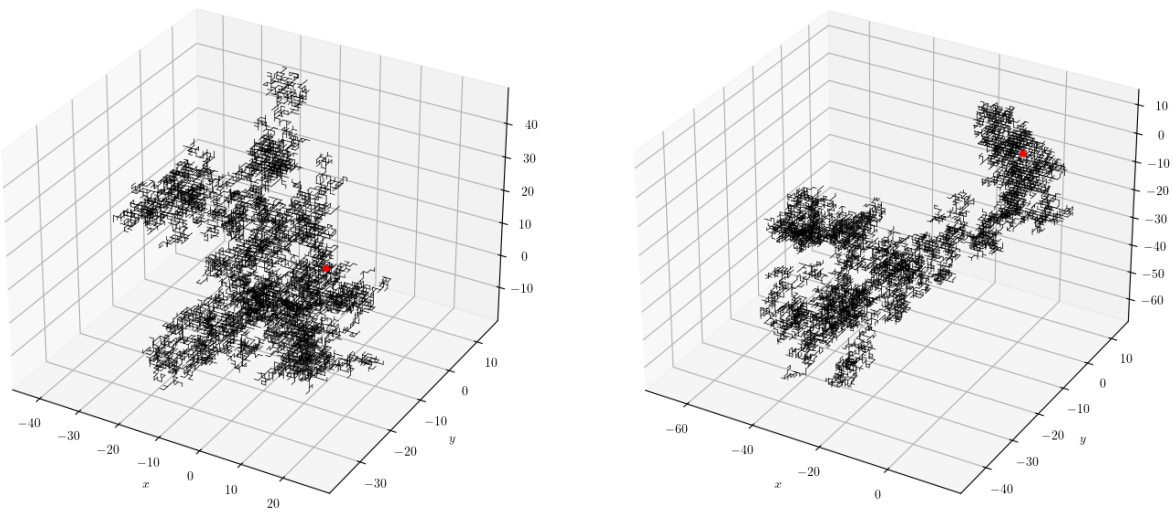

Figure 3: Simulation of invasion percolation in $\mathbb{Z}^2$ for 10^4 steps.

visualize only 10^4 steps, with the point marked in red is the origin $0 \in \mathbb{Z}^3$ where the invasion percolation process begins, and the black lines are the edges that have been included into the invaded region, *i.e.*, C_{10^4}. Again, we can see how the invasion percolation process differs a lot with different random seeds in its major invading direction and fractal shape.

Also, with the same setup as for $\mathbb{Z}^2$, we validate that the distribution of $G_{\mathbb{Z}^3}(0,x)$ follows a power law with respect to $|x|$. As shown in Figure 4, the distribution of $G_{\mathbb{Z}^3}(0,x)$ with $0 < |x| \leq 100$ respects the power law $G_{\mathbb{Z}^3}(0,x) \sim |x|^{-\alpha}$, with $\alpha \approx 0.61$ by simulation. This

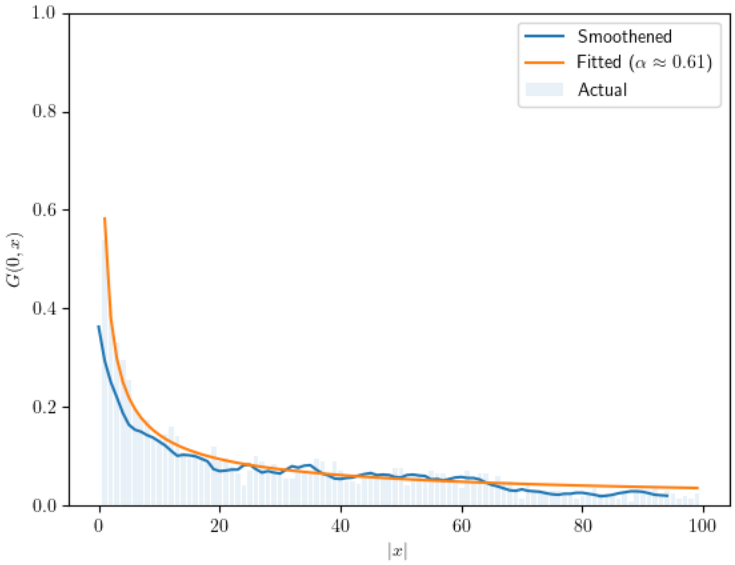

Figure 4: Distribution of $G_{\mathbb{Z}^3}(0,x)$ with 10^6 steps of invasion percolation in $\mathbb{Z}^3$, with each probability $G_{\mathbb{Z}^3}(0,x)$ computed from 300 independent samples.

is significantly larger thatn the $\alpha \approx 0.13$ in $\mathbb{Z}^2$, which is consistent with the intuition since the invasion percolation process in $\mathbb{Z}^3$ has more freedom of exploring the additional dimension compared with $\mathbb{Z}^2$, thus leading to lower probability of invading in a certain direction away from the origin, and thus faster decay of $G_{\mathbb{Z}^3}(0,x)$ compared with $G_{\mathbb{Z}^2}(0,x)$.

Comparison between $G_{\mathbb{Z}^2}(0,x)$ and $G_{\mathbb{Z}^3}(0,x)$. We can finally compare $G_{\mathbb{Z}^2}(0,x)$ and $G_{\mathbb{Z}^3}(0,x)$ for the same $|x|$, which is the missing component in proving $\mathbf{P}_\infty(p_c) = 0$ for $\mathbb{Z}^3$. We extend the comparsion to $|x| \leq 1000$ (in constrast to $|x| \leq 100$ in previous analyses) to obtain a more comprehensive result. As we can see in Figure 5, $G_{\mathbb{Z}^2}(0,x)$ dominates $G_{\mathbb{Z}^3}(0,x)$ for all $0 < |x| \leq 1000$, and the trend indicates that $G_{\mathbb{Z}^2}(0,x)$ will continue to dominate $G_{\mathbb{Z}^3}(0,x)$ for larger $|x|$ as well. Extending from previous power law conclusions that $G_{\mathbb{Z}^2}(0,x) \sim |x|^{-0.13}$ and $G_{\mathbb{Z}^3}(0,x) \sim |x|^{-0.61}$ to larger $|x|$ assuming more steps of invasion gives the same result. Hence, we have the following important observation.

Observation 3.1. $G_{\mathbb{Z}^3}(0,x) \leq G_{\mathbb{Z}^2}(0,x)$ for all $|x|$. This is not rigorously proven yet, but is supported by the simulation results in Figure 5.

Remark 3.2. $\mathbf{P}_\infty(p_c) = 0$ in $\mathbb{Z}^3$, *i.e.*, there exists an infinite cluster in a critical Bernoulli percolation in $\mathbb{Z}^3$ with probability zero. The proof we provide is, however, not rigorous because it is built on top of Observation 3.1 which is in turn simulation-based.

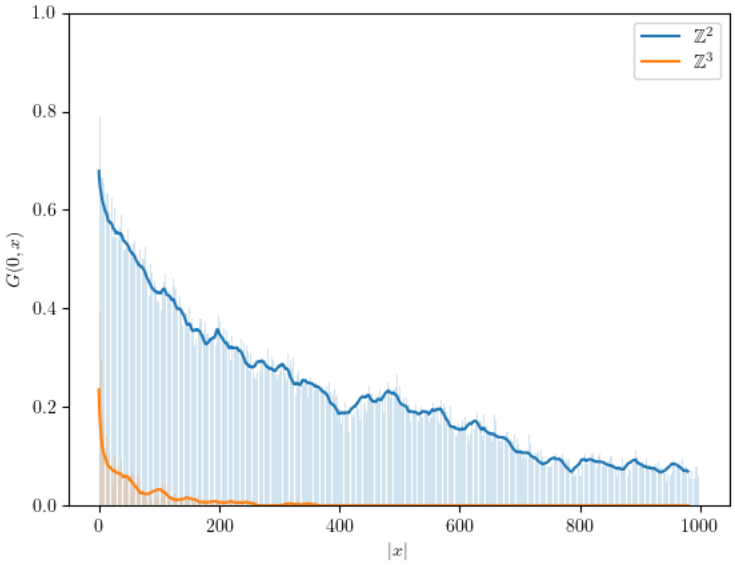

Figure 5: Comparing $G_{\mathbb{Z}^2}(0,x)$ and $G_{\mathbb{Z}^3}(0,x)$ for the same $|x|$, with 10^6 steps of invasion percolation in $\mathbb{Z}^2$ and $\mathbb{Z}^3$, respectively. Each probability $G_{\mathbb{Z}^2}(0,x)$ and $G_{\mathbb{Z}^3}(0,x)$ is computed from 300 independent samples.

Proof. This is a direct consequence of Observation 3.1 and the fact that $\mathbf{P}_\infty(p_c) = 0$ in $\mathbb{Z}^2$ [1]. By Theorem 2.3, since $\mathbf{P}_\infty(p_c) = 0$ in $\mathbb{Z}^2$, we have that $G_{\mathbb{Z}^2}(0,x) \to 0$ as $|x| \to \infty$. By Observation 3.1, we have that $G_{\mathbb{Z}^3}(0,x) \leq G_{\mathbb{Z}^2}(0,x)$ for all $|x|$ (and in particular as $|x| \to \infty$), and thus $G_{\mathbb{Z}^3}(0,x) \to 0$ as $|x| \to \infty$. By Theorem 2.3 again but the reverse direction, we can conclude that $\mathbf{P}_\infty(p_c) = 0$ in $\mathbb{Z}^3$. □

3.2. Invasion Percolation in Slabs

Slabs are often used in the study of invasion percolation. For instance in $\mathbb{Z}^d$, one might be interested in the behavior of invading a certain hyperplane $\{\, x \in \mathbb{Z}^d;\ x_d = N \,\}$, which can then be used in proofs for the whole of $\mathbb{Z}^d$. In the process of invading a certain hyperplane, it is common to divide the process into serveral steps, each step being the invasion across a slab with a certain thickness subject to translation in the d-th dimension. In order to investigate more fine-grained details of the invasion percolation process, especially when extending from $\mathbb{Z}^2$ to $\mathbb{Z}^3$, in this section we will further simulate the invasion percolation process in slabs

$$\mathbb{Z}_{2,l} := \mathbb{Z}^2 \times \{\, x_3 \in \mathbb{Z};\ |\, x_3| \leq l \,\}, \qquad l \in \mathbb{Z}, \tag{23}$$

Algorithm 2 Simulation of invasion percolation in slabs $\mathbb{Z}_{d-1,l}$.

1: $x_0 \leftarrow (0, \ldots, 0)$; `// Initial vertex`
2: $S_0 \leftarrow \{ (0, \ldots, 0) \}$; `// Set of invaded vertices`
3: $C_0 \leftarrow \varnothing$; `// Set of invaded edges`
4: $C'_0 \leftarrow \varnothing$; `// Set of candidate edges for the next invasion step`
5: **while** $i = 1, 2, \ldots$ **do**
6: $\quad C'_i \leftarrow C'_{i-1}$;
7: $\quad$ **for** $e \in \{$edges with x_{i-1} as one of its endpoints$\} \setminus C_{i-1} \setminus C'_{i-1}$ **do**
8: $\quad\quad$ `/* Avoid invading outside the slab */`
9: $\quad\quad$ **if** the other endpoint x_e of e has $|x_e| > l$ **then**
10: $\quad\quad\quad$ Pick a weight uniformly at random in $[0, 1]$ and assign it to e;
11: $\quad\quad\quad$ $C'_i \leftarrow C'_i \cup \{ e \}$;
12: $\quad\quad$ **end if**
13: $\quad$ **end for**
14: $\quad$ Pick $e_i \in C'_i$ with the smallest weight w_i;
15: $\quad$ $C_i \leftarrow C_{i-1} \cup \{ e_i \}$;
16: $\quad$ $C'_i \leftarrow C'_i \setminus \{ e_i \}$;
17: $\quad$ $x_i \leftarrow$ the other endpoint of e_i;
18: **end while**

for $l = 5, 10, 20$ and compare the results with $\mathbb{Z}^2$ and $\mathbb{Z}^3$. In order to limit the invasion percolation process within slabs, we first introduce a modified algorithm for simulating

$$\mathbb{Z}_{d-1,l} := \mathbb{Z}^{d-1} \times \{ x_d \in \mathbb{Z};\ |x_d| \leq l \}, \qquad l \in \mathbb{Z}, \tag{24}$$

as in Algorithm 2. The main difference with the basic version for $\mathbb{Z}^d$ is that, we do not add any edge that is not within the slab to the set of candidate edges C'_n (Line 9), and thus by edges beyond the slab boundaries will not be invaded in the process.

We use the same setup as in the previous section, with 10^6 steps of simulation and 300 independent samples for computing each value of $G_{\mathbb{Z}_{2,l}}(0, x)$. See Appendix A for visualizations of the first 10^4 steps of the invasion percolation process for each of $l = 5, 10, 20$, also as a validation of the correctness of the modified algorithm. We simulate the invasion percolation process in slabs $\mathbb{Z}_{2,l}$ for $l = 5, 10, 20$, for which the results are plotted in Figure 6. As we can see, the probability of invasion $G_{\mathbb{Z}_{2,l}}(0, x)$ decreases with as l increases, which is consistent with the intuition. Indeed, the smaller the slab width, the more constrained the invasion percolation process is on the third dimension of the space. This means that the invasion percolation process would be pushed to invade more in the first two dimensions, leading to higher $G_{\mathbb{Z}_{2,l}}(0, x)$ where we note that x is

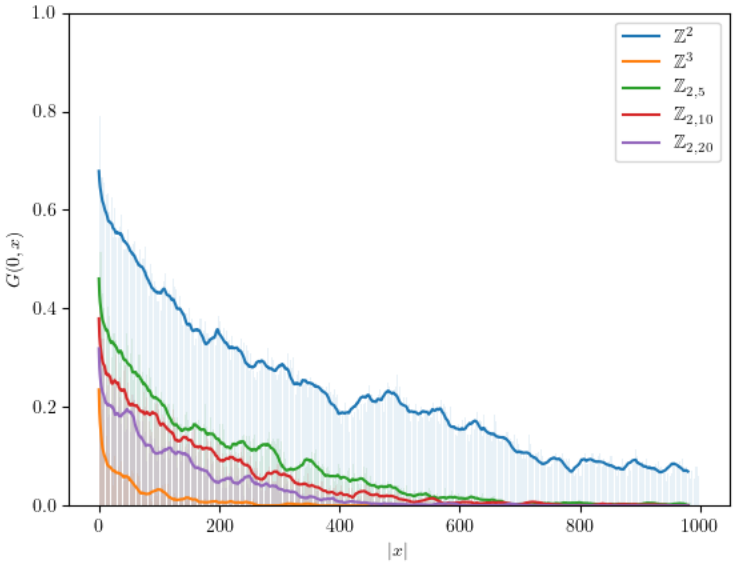

Figure 6: Comparing $G_{\mathbb{Z}_{2,l}}(0,x)$ among $l = 5, 10, 20$, as well as with $G_{\mathbb{Z}^2}(0,x)$ and $G_{\mathbb{Z}^3}(0,x)$ for the same $|x|$, with 10^6 steps of invasion percolation in each space. Each probability $G(0,x)$ is computed from 300 independent samples.

taken such that $x_3 = 0$. Similarly, we can see that

$$G_{\mathbb{Z}^2}(0,x) \leq G_{\mathbb{Z}_{2,l}}(0,x) \leq G_{\mathbb{Z}^3}(0,x), \qquad \forall |x| \leq 1000,\ l = 5, 10, 20. \tag{25}$$

With the same analyses as in the previous section, this observation should also naturally extend to arbitrary $|x|$ and $l \in \mathbb{Z}$. To understand this result, we may interpret $\mathbb{Z}^2$ as $\mathbb{Z}_{2,0}$, *i.e.*, a slab with zero width, and $\mathbb{Z}^3$ as $\mathbb{Z}_{2,\infty}$, *i.e.*, a slab with infinite width in the third dimension. Then the result would be consistent with the previous observation that $G_{\mathbb{Z}_{2,l}}(0,x)$ decreases with as l increases.

3.3. Supercritical Bernoulli Percolation in $\mathbb{Z}^d$

In this section, we will provide a brief sketch proof of a theorem regarding the supercritical Bernoulli percolation in $\mathbb{Z}^d$ for $d \geq 3$ [6]. This is not directly related to our goal of investigating the critical behavior of Bernoulli percolation in $\mathbb{Z}^3$, but it is an interesting result that (1) analyzes Bernoulli percolation via the invasion percolation process, (2) utilizes slabs in $\mathbb{Z}^d$ for the proof (though with a slightly different definition as in the previous section), and (3) to some extent, implies why the rigorous proof cannot be intuitively extended to critical Bernoulli percolation in $\mathbb{Z}^d$, as a partial evidence of the rationality of analyzing via simulation in this paper.

Define the slabs

$$L_l := \mathbb{Z}^{d-1} \times \{\, 0, 1, \ldots, l \,\}\,. \tag{26}$$

Note that we are ignoring the dimension d in this notation since throughout the rest of this section, we will always consider L_l as slabs in $\mathbb{Z}^d$. Let $\rho(l,p)$ be the probability that there exists an infinite cluster in a Bernoulli percolation with density p in L_l, analogous to $\mathbf{P}_\infty(p)$ in $\mathbb{Z}^d$. Define also p_c^l as the critical point as defined in (3), but with $\mathbf{P}_\infty(\cdot)$ in $\mathbb{Z}^d$ replaced by $\rho(l,\cdot)$ in L_l. Then the following theorem holds.

Theorem 3.3 (Chayes et al. [6])**.** Suppose $d \geq 3$ and $p > p_c^l$ for some $l \geq 0$. Then for any $N \geq 0$, we have that

$$\mathbb{P}((T_N < \infty) \wedge (w_n > p \text{ for some } n > T_N)) < (1 - \rho(l,p))^{N/(l+1)}, \tag{27}$$

where T_N is the time at which the invasion percolation process first invades the hyperplane $\{\, x \in \mathbb{Z}^d;\ x_d = N \,\}$.

Remark 3.4. This theorem implies that the probability of invading a hyperplane in finite steps while invading some vertex with weight larger then p after that step vanishes as the hyperplane is further away from the origin. As a direct consequence, it can be shown that the probability of $x \in \mathbb{Z}^d$ being included in a finite cluster around the origin is decays exponentially with respect to $|x|$, in a supercritical Bernoulli percolation with density $p > p_c^\infty$. We will present the original proof in [6] with slightly more details, which shall demonstrate how slabs can be useful in such proofs and why the proof cannot be intuitively extended to the critical cases.

Proof of Theorem 3.3 (Chayes et al. [6]). Let

$$L_l^j := \mathbb{Z}^{d-1} \times \{\, (j-1)(l+1), (j-1)(l+1)+1, \ldots, j(l+1)-1 \,\}\,, \qquad j \in \mathbb{Z}, \tag{28}$$

the slab translated by $(j-1)(l+1)$ of L_l. Denote by ψ_l^j the first time that the invasion percolation process invades L_l^j and by ξ_l^j the vertex at which this invasion occurs. Construct the invasion percolation within each slab L_l^j with the percolation cluster method with density p. The weights of edges between two slabs can be assigned when needed, according to the definition used in the simulation algorithms. Furthermore, when choosing the configurations ω_l^j within the slabs L_l^j in

the percolation cluster method, we can first choose a configuration in L_l and then shift the origin to ξ_l^j to obtain the configuration in L_l^j.

Now let $E_l^j(p)$ be the event that ξ_l^j is included in an infinite cluster. If $T_N < \infty$ for some $N \geq 0$, then $\psi_l^{N'} < \infty$ for some any $N' \geq 0$ such that $(N'-1)(l+1) \leq N$. Indeed, if we can invade the hyperplane $\{ x \in \mathbb{Z}^d;\ x_d = N \}$ in finite steps, then we can surely invade any slab L_l^j that has not gone beyond the hyperplane in finite steps. Denote by $\overline{N}$ the largest integer N' that satisfies the condition above. If $\psi_l^{\overline{N}} < \infty$ and $E_l^j(p)$ occurs for some $1 \leq j \leq \overline{N}$ (say j'), then $w_n \leq p$ for all $n \geq \psi_l^{j'}$ and thus $w_n > p$ for some $n > \psi_l^{j'}$. To see this, if $\xi_l^{j'}$ is in an infinite cluster, then all steps of invasion after j' must invade open edges (which are edges with weights at most p according to the percolation cluster method), since otherwise the connected component will break at some finite point given that $\psi_l^{\overline{N}} < \infty$.

Taking the contrapositive of the argument above, $w_n > p$ for some $n \geq \psi_l^{j'}$ implies that $E_l^j(p)$ does not occur for any $1 \leq j \leq \overline{N}$. Also recall that $T_N < \infty$ implies that $\psi_l^{\overline{N}} < \infty$, hence we can write

$$\mathbb{P}((T_N < \infty) \wedge (w_n > p \text{ for some } n > T_N)) \leq \mathbb{P}\left((\psi_l^{\overline{N}} < \infty) \wedge \bigcap_{j=1}^{\overline{N}} (E_l^j(p))^{\complement} \right). \tag{29}$$

Note that $E_l^{\overline{N}}(p)$ occurs with probability $\rho(l,p)$, *i.e.*, that there exists an infinite cluster in L_l. Moreover, it is independent of the events $E_l^j(p)$, $1 \leq j \leq \overline{N}-1$. Hence, we have that

$$\begin{aligned}
&\mathbb{P}\left((\psi_l^{\overline{N}} < \infty) \wedge \bigcap_{j=1}^{\overline{N}} (E_l^j(p))^{\complement} \right) \\
&\quad = (1-\rho(l,p))\mathbb{P}\left((\psi_l^{\overline{N}} < \infty) \wedge \bigcap_{j=1}^{\overline{N}-1} (E_l^j(p))^{\complement} \right) \\
&\quad \leq (1-\rho(l,p))\mathbb{P}\left((\psi_l^{\overline{N}-1} < \infty) \wedge \bigcap_{j=1}^{\overline{N}-1} (E_l^j(p))^{\complement} \right),
\end{aligned} \tag{30}$$

where the inequality is because the invasion percolation process must first cross the slab $L_l^{\overline{N}-1}$ (and thus invading it) before invading $L_l^{\overline{N}}$ since it starts from the origin. Note in addition that

$$\mathbb{P}((\psi_l^1 < \infty) \wedge (E_l^1(p))^{\complement}) = \mathbb{P}((E_l^1(p))^{\complement}) = 1 - \rho(l,p), \tag{31}$$

because ψ_l^1 is the first time that the invasion percolation process invades $L_l^1 = L_l$ which is finite

for sure. Hence by mathematical induction, we can obtain that

$$\mathbb{P}\left((\psi_l^{\overline{N}} < \infty) \wedge \bigcap_{j=1}^{\overline{N}} (E_l^j(p))^{\complement}\right) \leq (1-\rho(l,p))^{\overline{N}}. \tag{32}$$

Now recall (29) to complete the proof. □

This result cannot be intuitively extended to critical Bernoulli percolation because the conclusion relies on nonzero probability $\rho(l,p)$ to be meaningful (otherwise we will be claiming that a probability is at most one which does not provide any information). Recall that $\rho(l,p)$ is the probability that there exists an infinite cluster in a Bernoulli percolation with density p in L_l, and only when $p > p_c^l$ can this be possible by its definition.

4. Fractal Dimension of Invasion Percolation on $\mathbb{Z}^3$

In this section, we will numerically estimate the fractal dimension of the invaded region of an invasion percolation process in $\mathbb{Z}^3$. We will begin by introducing the concept of fractal geometry and the box-counting method for estimating fractal dimensions. Subsequently, we will describe the algorithm for computing the fractal dimension of invasion percolation clusters and present the results of our numerical analysis. By characterizing the geometric properties of invasion percolation clusters, we aim to gain insights into the spatial complexity and self-similarity of the invaded regions of invasion percolations in $\mathbb{Z}^3$.

Box-counting dimension. Fractal geometry serves as a robust analytical framework for dissecting the complex structures inherent in natural and synthetic systems. Fractals, characterized by their self-similarity and non-integer dimensions, offer a profound departure from traditional Euclidean geometry, allowing for a deeper understanding of spatial intricacies across various scales. In the context of invasion percolation, existing work has already demonstrated the fractal nature of invaded region of an invation percolation [9]. The application of fractal geometry offers a precise method for delineating the spatial characteristics of the invaded regions.

The *box-counting dimension*, also known as the Minkowski-Bouligand dimension, is a fundamental measure of the fractal dimension of a set. It is defined as

$$\dim(S) := \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)}, \tag{33}$$

where $N(\epsilon)$ is the number of boxes of side length ϵ required to cover the set S. The box-counting dimension provides a quantitative measure of the space-filling properties of a set, while being computationally tractable for estimating the fractal dimension of complex structures such as the invaded regions of invasion percolation processes.

Algorithm. Create a three-dimensional array of size $(2l+1)^3$, initialized with zeros, where l is chosen in advance. This is a three-dimensional cube with length $2l+1$, which represents $\{\, x \in \mathbb{Z}^3;\ |\, x_i| \leq l,\ i = 1, 2, 3 \,\}$. For each vertex $x \in S_n$ invaded by the invasion percolation process, set the corresponding entry in the array to one if within the cube. This way, we can easily determine the positions of the invaded vertices, thus the invaded region. Next, we determine the size of the boxes to cover the invaded region with. We start with a box of size 2^m where m is the smallest integer that allows covering the entire invaded region with only one box. We then gradually reduce the size of the box until the box is of size 2. For each box size, we count the number of boxes required to cover the invaded region. Finally, we compute the box-counting dimension by fitting a linear regression to the data points $(\ln(1/\epsilon), \ln N(\epsilon))$, according to the definition of $\dim(S)$. The implementation of the algorithm is based on that introduced in [10], extending it to three dimensions.

Results. Same as in Section 3, we simulate 10^6 steps of invasion due to technical constraints. We choose $l = 100$ and compute the box-counting dimension of the invaded region. The reason for restricting within $l = 100$ is that 10^6 steps of invasion is not sufficient to invade the region beyond $l = 100$ to a steady state. We have repeated the computation for 10 independent runs of the invasion percolation process with different random seeds, and discovered that $l = 100$ is a reasonable choice where the result stabilizes among different runs while still being large enough to capture the fractal nature of the invaded region. Figure 7 shows an overview of the shape of the invaded region within the finite cube.

We compute $\ln N(\epsilon)$ and measure the ratios $\ln N(\epsilon)/\ln(1/\epsilon)$ for ϵ ranging from 128 (where a single box covers the whole invaded region) to 2 (the stopping criterion). Ignoring the $\epsilon = 128$ where $N(\epsilon) = 1$ and hence $\log N(\epsilon) = 0$, we perform the least-squares linear fit to the the obtained data points $(\ln(1/\epsilon), \ln N(\epsilon))$ and record the slope of the fitted line. As is shown in Figure 8, the fitted slope is approximately 2.627, which would be our estimate of the box-counting dimension of the invaded region of an invasion percolation process in $\mathbb{Z}^3$. This suggests that the invaded region has a complex and self-similar structure that does not fill up the three-dimensional space

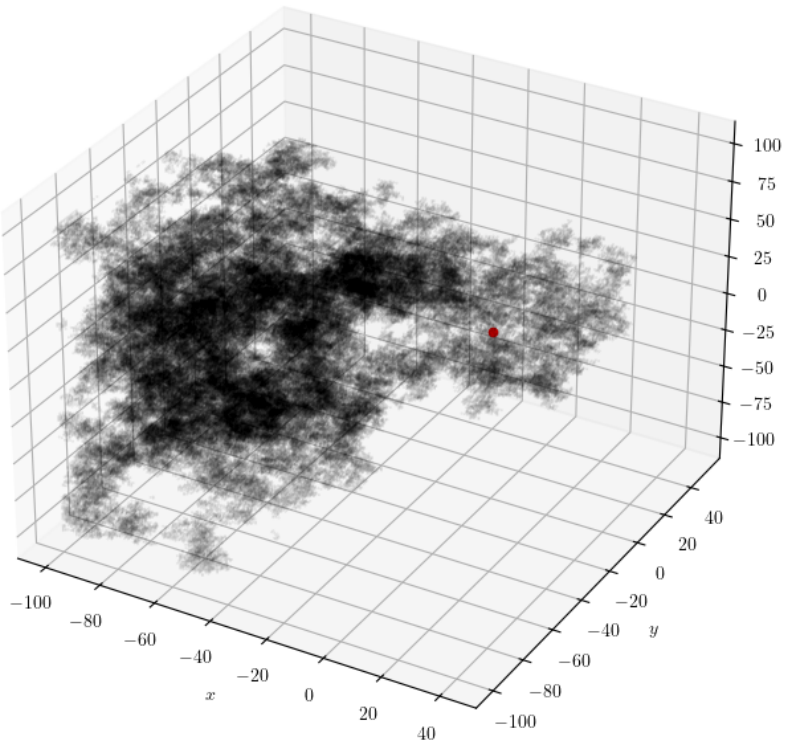

Figure 7: An invasion percolation process in $\mathbb{Z}^3$ with 10^6 steps, visualized within a finite cube.

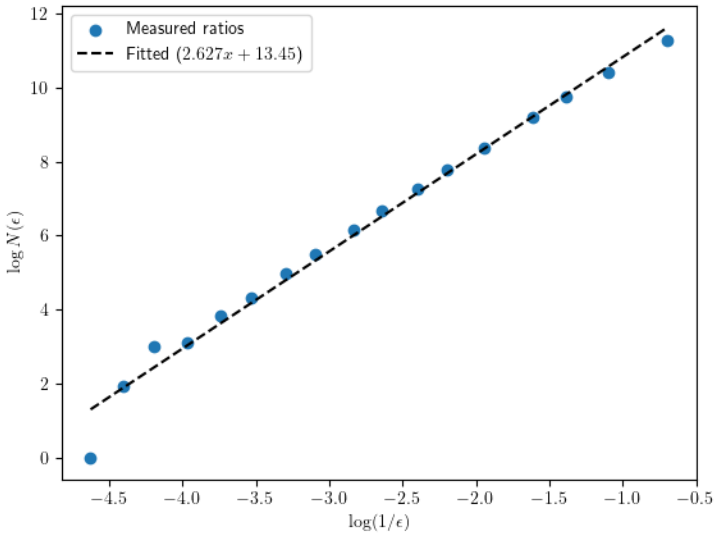

Figure 8: Fitting $(\ln(1/\epsilon), \ln N(\epsilon))$ of the invaded region of invasion percolation in $\mathbb{Z}^3$.

uniformly.

5. Conclusion

This paper has investigated the relationship between Bernoulli percolation and invasion percolation, based on which it has shown that the infinite cluster density $\mathbf{P}_\infty(p_c) = 0$ at the critical

point for Bernoulli bond percolation in $\mathbb{Z}^3$, extending the known result in $\mathbb{Z}^2$ with the help of computational simulation of the invasion percolation process. The most important simulation result is that $G_{\mathbb{Z}^3}(0,x)$ is dominated by $G_{\mathbb{Z}^2}(0,x)$ for all $|x|$, which is also the observation that contributed to the proof of $\mathbf{P}_\infty(p_c) = 0$. The paper has also analyzed invasion percolation in slabs that might be helpful for rigorous proofs of the results in future research. It has in addition examined the geometric properties of the invaded regions of invasion percolation by estimating their fractal dimensions, which is approximately 2.627 in $\mathbb{Z}^3$.

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A. Visualizing Invasion Percolations in Slabs $\mathbb{Z}_{2,l}$

This section presents the visualizations of invasion percolation in slabs $\mathbb{Z}_{2,l}$ for $l = 2, 5, 10, 20$ for 10^4 steps by Algorithm 2. Each of Figure 9, Figure 10, and Figure 11 presents three independent simulation results corresponding to three different random seeds used for assigning the edge weights. The semi-transparent surfaces represents the maximum and minimum values in the first two dimensions (unconstrained ones) that the invasion percolation process invaded within the first 10^4 steps. As we can see within each figure, the positions and shapes of the invaded regions differ a lot, as with $\mathbb{Z}^2$ and $\mathbb{Z}^3$ noted in Section 3.1. The more intriguing result comes from comparing the figures, where we can see that the invaded regions in the first two dimensions get smaller as the slab width l increases. This is consistent with our conclusion in Section 3.2 that $G_{\mathbb{Z}_{2,l}}$ decreases as l increases. This is also intuitively self-explanatory, as the invasion percolation process has more freedom to explore the third dimension as l increases, hence invading less in the first two dimensions.

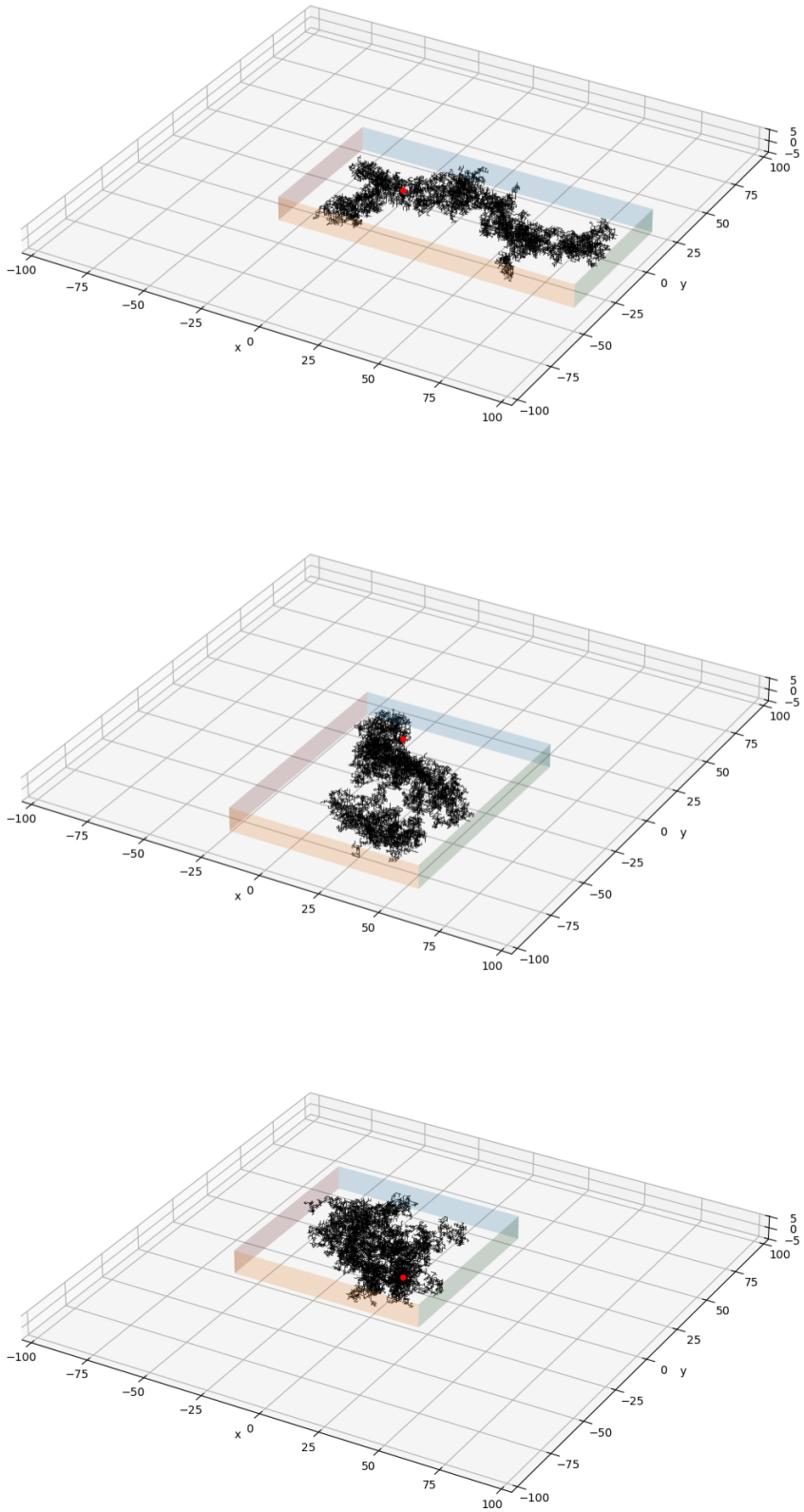

Figure 9: Simulation of invasion percolation in $\mathbb{Z}_{2,5}$ for 10^4 steps.

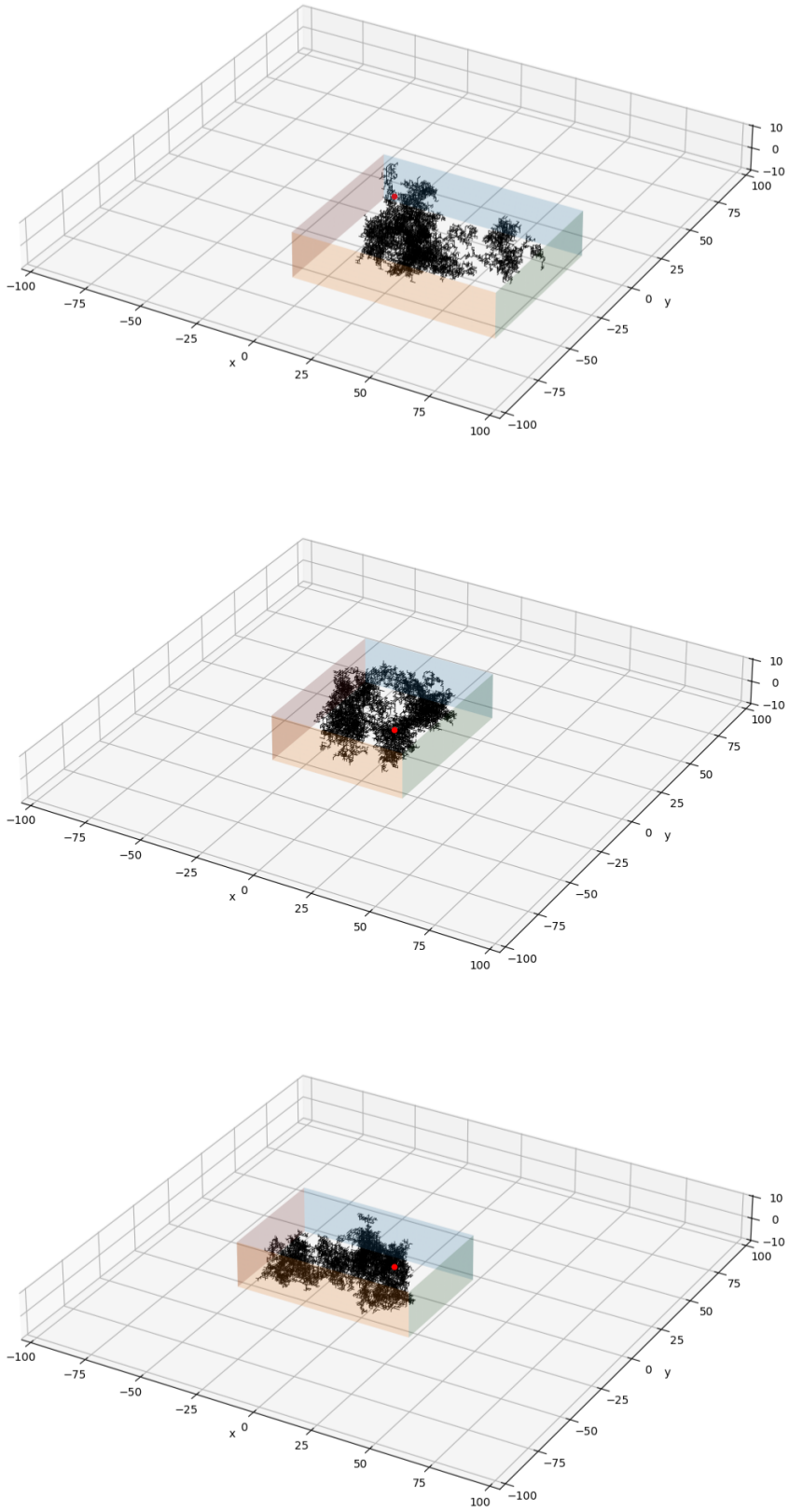

Figure 10: Simulation of invasion percolation in $\mathbb{Z}_{2,10}$ for 10^4 steps.

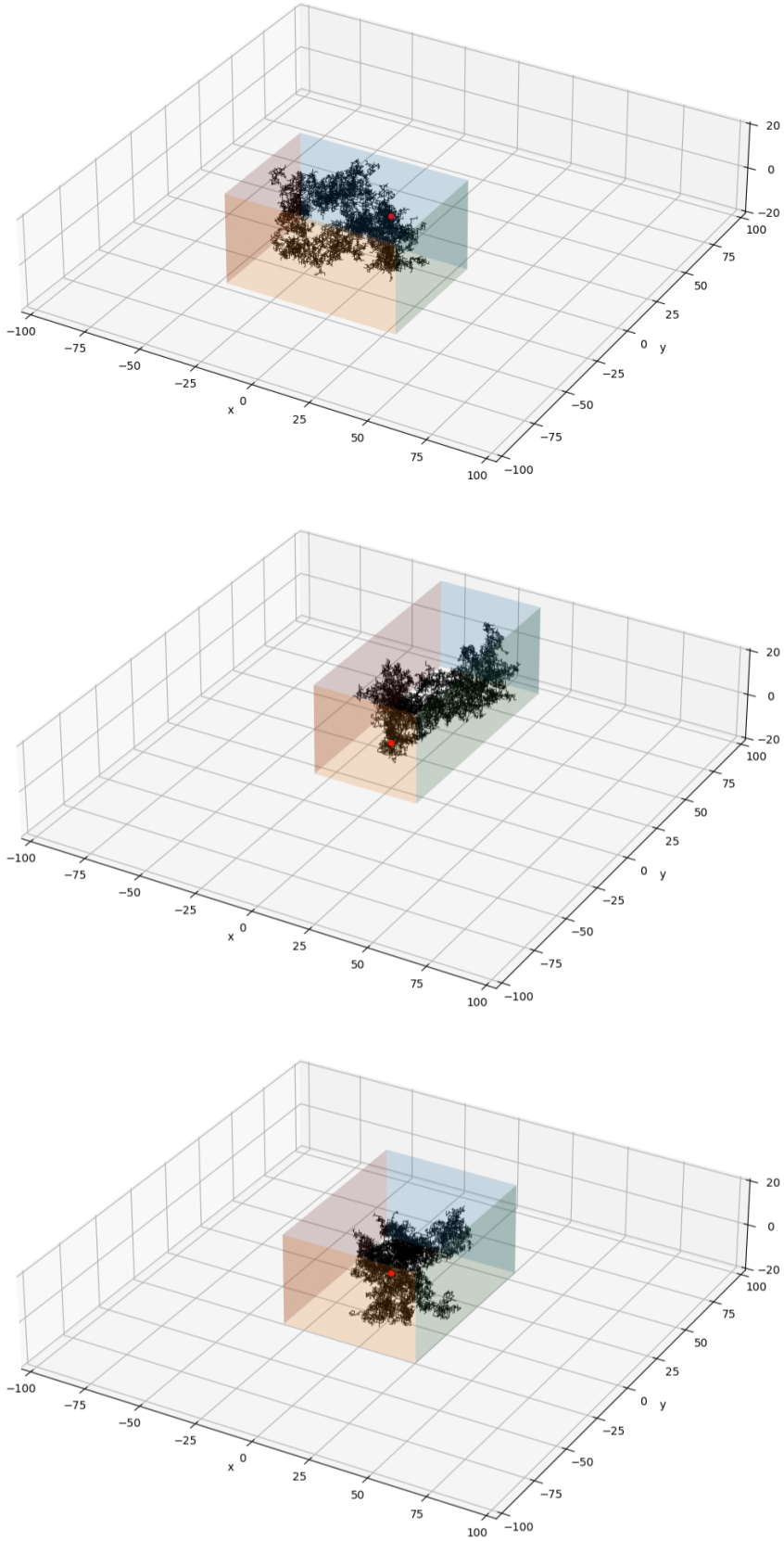

Figure 11: Simulation of invasion percolation in $\mathbb{Z}_{2,20}$ for 10^4 steps.